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Retirement Indexation in a Stochastic Model with Overlapping Generations and Endogenous Labour Supply*

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Abstract

Using a stochastic overlapping generations model with endogenous labour supply, this paper studies the design and performance of a policy rule for the retirement age in response to fertility and mortality shocks. Two main results are derived: First, to offset a change in the labour force the retirement age should adjust more than proportionally to the fertility change and, second, to be socially desirable the retirement age should be indexed less than proportionally to changes in life expectancy.

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Keywords: Retirement age, endogenous labour supply, overlapping generations, intergenerational risk sharing, method of undetermined coefficients.

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1 Introduction

The combination of fertility slowdowns and increasing longevity in developed economies may have dramatic economic effects. The purpose of this paper is, first, to analyse the impact of fertility and mortality shocks on key macroeconomic variables and, second, to discuss the design of a policy rule that ensures a more equitable distribution profile of welfare across generations.

The dependency ratio is partly shaped by the "baby-boom" phenomenon of the 1940-50s followed by a baby-bust in the 1970-80s (see, e.g., Bongaarts, 1998, and IMF, 2004). As a consequence, the labour force is currently shrinking and the number of retirees is increasing. An important question is, therefore, how to design a policy rule for the retirement age that effectively counteracts this decline in the labour force. Longevity of the elderly, which is expected to increase permanently (see, e.g., Oeppen and Vaupel, 2002; UN, 2004), is also increasing the dependency ratio. There is a large body of literature on the subject of demographic change and viability of social security arrangements (see e.g. Auerbach and Lee, 2001; Campbell and Feldstein, 2001; and Cutler et al., 1990), but this is not the focus of this paper. It will suffice to note that the period of time during which retirees collect pension benefits increases for two reasons: retirees are expected to live longer, and they tend to retire earlier – partly because they attach a higher weight to leisure in line with increasing economic prosperity. As a result, the retirement period is extended at both ends.

In order to devise appropriate policy responses, the dynamics of these effects must be well understood. If, under existing welfare arrangements, it turns out that retirees lose more than workers, there may be a rationale for economic policy to redistribute. This could be achieved through higher wage taxes if retirees are hurt more than workers. However, since this would distort the incentives to work, it might not be the right thing to do. Furthermore, due to the permanent nature of the increases in longevity, a tax-smoothing strategy (Barro, 1979) may not be what is called for. If a temporary fertility decline was the only demographic change it would make more sense to raise and smooth taxes.

Since leisure is often considered a consumption-equivalent (normal) good, labour supply may fall in line with economic growth[1]. This will exacerbate the negative impact on labour supply following the decline in fertility. To offset these labour supply dynamics the incentive of workers to work more could be stimulated through lower income taxes, but that would be difficult in light of the discussion above. In addition, workers' inclination to retire earlier, financed by their own personal savings, is problematic to legislate against in liberal welfare states. In several countries, the reaction to these dilemmas has been to increase the statutory retirement age in order to retain workers in the labour force for a longer period. Against that, the aim of this paper is to analyse the *optimal* response of the retirement age as a policy instrument to deal with the changes in fertility and longevity.

The magnitudes of declines in fertility (and thus in labour forces) are well known. However, the changes in life expectancy are inherently uncertain, which warrants a stochastic approach. In the literature, stochastic population projec-

[1] This is a plausible explanation for the upward trend in early retirement and the increased demand for leisure during working life.

tions, in line with, e.g., Alho and Spencer (1985), have been combined with CGE models to produce stochastic trajectories of key economic variables (see, e.g., Fehr and Habermann, 2008). However, while useful in many respects, a drawback of such large-scale (black-box) simulations is that it may be very difficult to identify the seperate role of each variable and parameter. An alternative approach, as pursued in this paper, is to formulate a dynamic stochastic general equilibrium (DSGE) model. The advantage of deriving the economic effects of demographic shocks analytically is that the role of each model parameter can be identified more accurately. Furthermore, the derived analytical expressions can be calibrated, in order to underscore the role of the fundamental model properties using numerical simulations.

Building on these advantages of an analytical solution, this paper develops a DSGE model with overlapping generations (OLG). The novelty of the model is to incorporate the retirement age together with endogenous labour supply, in a way that emphasises both the *extensive* and the *intensive* margins of labour supply when tracing the effects on different generations from stochastic shocks to fertility and life expectancy. However, the fact that labour supply is endogenous makes the dynamic system much more complicated to solve analytically. The technical innovation of this paper is to apply the method of undetermined coefficients (Uhlig, 1999) in a way that enables me to solve for endogenous labour supply (and in principle for an unlimited number of additional state variables). While this solution method by now is standard in the RBC literature, this paper shows how the same method can be used in the context of stochastic OLG models.

Since the capital-labour ratio is endogenous in the model, wages and the interest rate are affected when demographic shocks appear[2]. For example, if fertility declines there will be a tendency for wages to increase. Regarding the increase in life expectancy, workers are expected to consume less, and to save and work more, in order to finance their longer retirement period. The net effect on retirees, on the other hand, is also ambiguous because, first, the interest rate falls, due to the decline in fertility, but then it increases, because of the increased labour supply induced by the increase in life expectancy. Based on this setup we find that workers and retirees are affected in different proportions by changes in fertility and life expectancy. We therefore ask whether this market allocation is fair compared to a socially optimal allocation, and we find that it does not correspond with the optimality conditions for welfare. This motivates me to consider the retirement age as an alternative policy instrument to provide a more equitable outcome across generations.

The impact on economic variables is presented in terms of analytical expressions, incorporating the changes in leisure, and it is feasible to derive the impacts on labour supply of changes in the retirement age, in fertility, and in life expectancy. This would not be feasible in a model with exogenous labour supply. Because of the closed form solution for economic responses to demographic changes, it is possible to isolate the retirement age for any change in fertility and life expectancy. For instance, if fertility fell by 1% and life expectancy increases by 1% we find that

[2] In fact, Kotlikoff, Smetters and Walliser (2001), Welch (1979) and Murphy and Welch (1992) find evidence that demographic changes affect factor prices, which emphazises the importance of a general equilibrium model with endogenous interest rates and wages.

the retirement age should increase by 0.25%. This is because workers bear a lower burden than retirees, and because workers tend to supply less labour when the retirement age increases[3]. In this context, an innovation of this paper is to derive the elasticity for leisure with respect to the retirement age, which we find to be positive. The elasticities of leisure with respect to fertility and life expectancy are both negative. These elasticities can be directly traced to the fundamental model properties and the underlying parameters. A second result is that, regardless of intergenerational welfare perspectives, the decline in the *effective* labour force can be offset by an increase in the retirement age if it is more than proportional to the fertility fall. This is because workers substitute for leisure when fertility falls and the retirement age increases. The optimal response of the retirement age to offset the decline in the labour force is found to be 1.1% for each fall in fertility of 1%.

The next section develops the model and the analytical solution method, while section 3 presents the solution for key variables in the market equilibrium in response to changes in fertility, life expectancy, and the retirement age. Section 4 then considers the policy option of changing the retirement age in accordance with intergenerational fairness. Finally, section 5 concludes and outlines an agenda for potential extensions of the research on this topic.

2 The Model

This section outlines a stochastic OLG model *ad modum* Bohn (2001). In that paper the length of the *retirement* period is incorporated into a model with *exogenous* labour supply. The model in this paper extends this structure by first *endogenising* labour supply and, second, by incorporating the length of *working* period, such that if its length changes so will the retirement age. We can then identify *three* components of effective labour supply: the exogenous extensive margin, the endogenous intensive margin, and the growth in the number of workers.

By making these extentions, all the results and policy implications in Bohn (2001) will also be modified. In fact, using Bohn's model it is not possible to analyse neither the policy choice of adjusting the retirement age in response to demographic changes, nor the effects on households' labour-leisure choices of changes in the retirement age. OLG models with endogenous *intensity* of labour supply are common in the literature, but a model that is also combined with changes in labour supply at the *extensive* margin has, to my knowledge, not previously been developed.

Below, we present the stochastic OLG model, consisting of the demographic structure, household behaviour, technology, resources, and the pension system. The last subsection presents the analytical solution method.

2.1 Demographics

Individuals are assumed to live for three periods: as children, adults and elderly, respectively, and individuals in each cohort are assumed to be identical. Children born in period t is denoted by N_t^c, where $N_t^c = b_t N_t^w$ and $b_t > 0$ is the birth rate.

[3]This latter argument could be interpreted as if early retirement is more frequent, or alternatively, as a decrease in working hours per week.

Adults are all assumed to work for the full length of period t and are denoted by N_t^w, while they are retireed during period $t+1$. The labour force grows by $N_t^w/N_{t-1}^w = 1 + n_t^w$, where $n_t^w \equiv \chi_t b_{t-1}$ is the (net) growth rate of the labour force and χ_t is the length of the working period[4]. Workers supply labour, L_t, elastically up to one unit: $\{1-l\} \in (0,1)$, where $L_t = (1-l_t)\, N_t^w$.

The aggregate adult lifetime, ϕ, is composed by the adult working and retirement periods, respectively, as illustrated in figure 1.

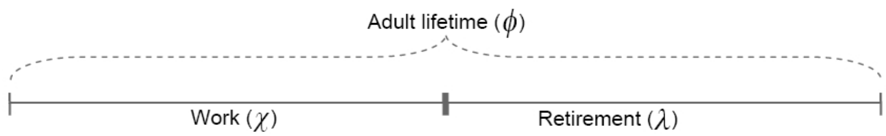

Figure 1. Aggregate adult lifetime: work and retirement

Specifically, ϕ comprises an expected term and an unexpected term, i.e. $\phi_t = \phi_{t-1}^e \phi_t^u$, where $\{\phi\} \in (0,2)$. The components in ϕ_t are assumed to be stochastic and identically and independently distributed, as are b_t and χ_t. The aggregate adult lifetime thus comprises the lengths of the working and retirement periods, from where the latter, $\lambda \in (0,1)$, is residually determined in (1).

$$\lambda_t = \phi_t - \chi_{t-1} \tag{1}$$

An increase in χ will therefore lead to a proportional decrease in the length of the retirement period, λ. Furthermore, changes in total length of life, ϕ, entirely impacts upon λ if χ remains constant. Based on this setup, we argue that an increase in χ can be interpreted as an increase in the retirement age.

2.2 Household Behaviour

Parents are assumed to make economic decisions about consumption on behalf of their children and themselves. A childhood period is conceptually necessary in this model in order to study a change in fertility in period $t-1$ that affects the size of the labour force in period t. However, an explicit formulation of the optimisation of parents' utility over their own consumption and that of their children is not necessarily important because the optimisation problem would merely relate first period consumption of the household to the weight that parents assign to consumption of their children in utility. This relation can be shown to enter into lifetime utility as a weight on first period consumption, $\rho_{1(b_t)} > 0$, that depends positively on the number of children, see Jensen and Jørgensen (2008)[5].

[4] If χ_t increases then workers must remain in the labour force for more "sub-periods", which is why the *net* growth rate of the labour force increases. Similarly, if fertility was low in the former period the current labour force decreases. The equivalent notation in a standard Diamond (1965) OLG model would just have a normalised length of working period ($\chi_t = 1$), so that the growth rate of the labour force is b_{t-1}, i.e. $N_t^w/N_{t-1}^w = 1 + b_{t-1}$.

[5] We assume, however, that a 1% increase in fertility would increase $\rho_{1(b_t)}$ by 1%, because parents need to provide more consumption to more children in the household. The elasticity of

Utility is assumed to feature homothetic preferences, and is defined over consumption and leisure as follows,

$$u_t = \chi_t \left[\rho_{1(b_t)} \ln c_{1t} + \gamma \ln \left(\frac{l_t}{\chi_t} \right) \right] + \rho_2 E_t \left[\lambda_{t+1} \ln c_{2t+1} \right] \tag{2}$$

where c_{1t} is first period consumption, c_{2t+1} is second period consumption, l_t is first period leisure and $\gamma > 0$ its relative weight, while $\rho_2 > -1$ is the discount rate on second period consumption. Retirees are assumed not to leave any bequests.

Second period consumption is scaled by the length of the retirement period, because the higher λ is the longer retirees can enjoy consumption. The same argument applies to the length of the first period, χ. However, if χ increases some of the "full-leisure" periods in retirement will be substituted by periods of both labour and leisure in the working period. This results in a decreasing lifetime leisure, the disutility of which must be accounted for relative to leisure. Consequently, we scale l_t by χ_t. If the retirement age increases, individuals can then account for the disutility of less lifetime leisure by increasing leisure[6]. Effective labour supply would initially rise by the full amount of the increase in the retirement age, but this effects will be counteracted if the disutility for workers of less lifetime leisure induces them to supply labour less intensively.

Budget restrictions on c_{1t} and c_{2t+1} are stated in (3) and (4), respectively.

$$\chi_t c_{1t} = (1 - \theta_t)(1 - l_t) w_t - S_t \tag{3}$$

$$c_{2t+1} = \frac{R_{t+1}}{\lambda_{t+1}} S_t + \beta_{t+1} (1 - l_{t+1}) w_{t+1} \tag{4}$$

In second period consumption the gross return to retirees' savings, R_t, is now scaled by the length of the retirement period. The length of second period of life has been incorporated in similar ways by both Bohn (2001) and Chakraborty (2004). However, in the former paper λ does not depend residually on the length of the working period, χ, and in the latter paper, λ is incorporated differently and denotes the length of total life and at the same time the discount rate, and Chakraborty also makes it endogenous to health expenditure. In the present paper, we also consider λ to be endogenous – depending on shocks to either the total length of adult life or to the length of working life, i.e. $\lambda = \phi - \chi$. In that way changes in the retirement age is seen to automatically affect the length of the retirement period, which could not be analysed by Bohn (2001) and Chakraborty (2004).

The intertemporal budget constraint (IBC) is derived by combining c_{1t} and c_{2t+1} over savings to yield:

$$\chi_t c_{1t} + \frac{\lambda_{t+1}}{R_{t+1}} c_{2t+1} + (1 - \theta_t) w_t l_t = (1 - \theta_t) w_t + \frac{\lambda_{t+1}}{R_{t+1}} \beta_{t+1} (1 - l_{t+1}) w_{t+1} \tag{5}$$

The variable w_t denotes the wage rate, S_t is savings, θ_t is the pension contribution rate, and β_t is the pension replacement rate.

$\rho_{1(b_t)}$ is therefore assumed to be equal to $\pi_{\rho_1(b)} = 1$, where π denotes the elasticity of the weight on first period consumption in utility with respect to the number of children in the household.

[6] This feature is a way to implicitly add second period leisure into the utility function without explicitly maximising with respect to l_{t+1}.

By maximising utility (2) subject to the IBC (5) the two first order conditions are derived: the Euler equation and the optimality condition for first period consumption and leisure in (6) and (7), respectively.

$$c_{1t} = \left(\frac{\rho_{1(b_t)}}{\rho_2}\right) \ E_t \left\{\frac{c_{2t+1}}{R_{t+1}}\right\} \tag{6}$$

$$\frac{l_t}{\chi_t} = \left(\frac{\gamma}{\rho_{1(b_t)}}\right) \ \frac{c_{1t}}{(1-\theta_t)\, w_t} \tag{7}$$

If fertility increases so does $\rho_{1(b_t)}$ in (6) and (7), and households prioritise additional consumption in the first period so that c_{1t} increases relative to c_{2t+1} and l_t. Note from (7) that if χ_t increases so must l_t because workers compensate for less lifetime leisure by increasing first period leisure. The uniqueness of this relationship illustrates that an increase in the retirement age would induce an increase in leisure and consequently a fall in the intensive margin of labour supply. This link emphasizes that the increasing effect on effective labour supply of a rise in the retirement age will be counteracted by the demand for leisure.

2.3 Resources and Social Security

Firms are assumed to produce output with capital and labour according to the assumed Cobb-Douglas technology,

$$Y_t = K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha}$$

where K_t is physical capital, productivity is A_t which is stochastic and grows at a rate, a_t, so $A_t = (1 + a_t)\, A_{t-1}$, where a_t is assumed identically and independently distributed. The wage rate and the return to capital are obtained through $w_t\,(k_t) = f\,(k_t) - k_t f'\,(k_t)$, and $R_t\,(k_t) = f'\,(k_t)$, where the capital-labour ratio is defined over growth rates as $k_{t-1} \equiv K_t / \left(A_{t-1} L_{t-1}\right)$. Capital is accumulated through workers' savings, i.e. $K_{t+1} = N_t^w S_t$, and by assuming that firms are identical, and that capital fully depreciates over one generational period, the resource constraint of the economy is:

$$Y_t - K_{t+1} = \chi_t N_t^w c_{1t} + \lambda_t N_{t-1}^w c_{2t} \tag{8}$$

The PAYG system is defined as follows:

$$\lambda_t \beta_t N_{t-1}^w \left(1 - l_t\right) w_t = \theta_t N_t^w \left(1 - l_t\right) w_t \tag{9}$$

The PAYG system can feature both defined (fixed) benefits (DB) and defined contributions (DC) schemes, since neither β nor θ are necessarily fixed. For instance, solving for a DB system yields (10).

$$\theta_t = \beta \left(\frac{\phi_t - \chi_{t-1}}{1 + n_t^w}\right) \tag{10}$$

Evidently, with the replacement rate held fixed, an increase in the working period, χ, leads to a lower contribution rate. Similarly, an increase in the total length of life calls for a higher contribution rate.

In this paper, the focus is not on pension reform in terms of changing the replacement and/or contribution rates more or less than they automatically change when the dependency ratio changes, hence our notion of a *passive* pension system. This completes the presentation of the stochastic OLG model. In the next section a method to solve the model analytically is presented.

2.4 The solution method

We are interested in an analytical closed form solution of the model that provides intuition on the impact on economic variables when fertility and life expectancy change. The advantage of this analytical approach is that changes in each economic variable can be traced to the fundamental model properties and underlying parameters. Furthermore, the exact policy rules that will generate optimal distributions of welfare can be derived analytically, and their implications can be traced in detail across different generations.

The method of undetermined coefficients is used to obtain this analytical solution for the recursive equilibrium law of motion - charaterised by providing the solution in terms of analytical elasticities of economic variables with respect to demographic shocks. By adopting this approach the non-linear stochastic OLG model is replaced by a log-linearised approximate model with variables (denoted with "hats") stated in terms of percentage deviations from their steady state values. We adopt a version of the method of undetermined coefficients that relies on Uhlig (1999), which we extend to account for expected changes in life expectancy[7]. We refer to the details of our extension of the solution method in Appendix A[8].

All endogenous variables from the log-linearised model, $\widehat{e}_t \in \{\widehat{k}_t, \widehat{c}_{1t}, \widehat{c}_{2t}, \widehat{l}_t, \widehat{y}_t, \widehat{R}_t, \widehat{w}_t, \widehat{\theta}_t\}$, are written as linear functions of a vector of endogenous and exogenous state variables, respectively. The vector of endogenous state variables is $\widehat{x}_t \in \{\widehat{k}_t, \widehat{c}_{2t}\}$, the vector of endogenous non-state variables is $\widehat{v}_t \in \{\widehat{c}_{1t}, \widehat{l}_t, \widehat{y}_t, \widehat{R}_t, \widehat{w}_t, \widehat{\theta}_t\}$, and the vector of exogenous state variables (including the demographic shocks to fertility, $\widehat{b}_{t-1}$, and life expectancy, $\widehat{\phi}_t^e$) is $\widehat{z}_t \in \{\widehat{\chi}_{t-1}, \widehat{\chi}_t, \widehat{a}_t, \widehat{b}_{t-1}, \widehat{b}_t, \widehat{\phi}_{t-1}^e, \widehat{\phi}_t^e, \widehat{\phi}_t^u\}$. The recursive equilibrium is characterised by a conjectured linear law of motion between endogenous variables in the vector $\widehat{e}_t$, and state variables in the vectors $\widehat{v}_t$ and $\widehat{z}_t$. As an example of how a given endogenous variable is determined we illustrate the linear law of motion for leisure, $\widehat{l}_t$, in (11), where e.g. $\pi_{l\chi}$ denotes the elasticity (π) of leisure (l) with respect to the retirement age (χ). All endogenous variables in $\widehat{e}_t$ can be expressed in this fashion.

$$\begin{aligned}
\widehat{l}_t &= \pi_{lk}\widehat{k}_{t-1} + \pi_{lc2}\widehat{c}_{2t-1} + \pi_{l\chi 1}\widehat{\chi}_{t-1} + \pi_{l\chi}\widehat{\chi}_t \\
&\quad + \pi_{la}\widehat{a}_t + \pi_{lb1}\widehat{b}_{t-1} + \pi_{lb}\widehat{b}_t + \pi_{l\phi e1}\widehat{\phi}_{t-1}^e + \pi_{l\phi e}\widehat{\phi}_t^e + \pi_{l\phi u}\widehat{\phi}_t^u
\end{aligned} \tag{11}$$

[7] If current workers expect their lives to be longer this change will ultimately take place at the end of their retirement period, i.e. we analyse an exogenous shock to life expectancy that is expected to take place in the next period.

[8] A less advanced version of the method of undetermined coefficients (based on only one state variable) was first applied on OLG models by Andersen (1996, 2001) and Bohn (1998, 2001) in models without endogenous labour supply and without a retirement age, and by Jensen and Jørgensen (2007) in a model that incorporates the retirement age but still without endogenous labour supply.

Changes in e.g. life expectancy is inherently stochastic, and with this solution method it is feasible to specify distributions for the stochastic innovations in life expectancy and simulate the impulse responses on e.g. leisure. One advantage with this solution method is, however, that the impact on leisure is stated in terms of an elasticity, $\pi_{l\phi e}$, the size of which naturally assumes a 1% shock to life expectancy, $\widehat{\phi}_t^e$. So instead of evaluating the impact on leisure of some pre-specified distribution of stochastic innovations for life expectancy we simply ask the question: "*how will leisure change if there was suddenly an increase in life expectancy of 1%*". In this terminology we basically make comparative statics with an otherwise stochastic model (see Uhlig, 1999; Campbell, 1994)[9]. Note that the size of the stochastic shock could, of course, be any value from a pre-specified distribution.

The purpose of the following section is to interpret these elasticities, both intuitively and numerically, and to employ them in policy reflections on intergenerational welfare. This involves calibrating the model using what we believe are realistic parameter values, as listed in Appendix B, and simulating the model using a Matlab routine (available upon request).

3 Market equilibrium and demographic shocks

How do demographic changes affect the welfare of different generations? To address this question, we interpret the elasticities of macroeconomic variables with respect to demographic changes. Attention is restricted to *three* shocks: first, a shock to the lagged birth rate, $\widehat{b}_{t-1}$. Second, a shock to life expectancy, $\widehat{\phi}_t^e$. Third, a change in the retirement age, $\widehat{\chi}_t$. We analyse the change in the retirement age as a stochastic shock in order to derive the effects it entails. In section 4, on the other hand, it is assumed that the retirement age is under government control which facilitates the use of the retirement age as a policy instrument[10]. We furthermore assume a PAYG system with fixed benefits for the remaining sections of the paper, but we do, however, provide perspectives to a DC system. As to the economic effects, the focus is on consumption possibilities for workers ($\widehat{c}_{1t}$) and retirees ($\widehat{c}_{2t}$), respectively, and on leisure for workers ($\widehat{l}_t$)[11].

3.1 A shock to fertility

Industrialised countries are currently experiencing that a historically low numbers of young workers are entering the labour force, due to low fertility in the 1970-80s. How workers and retirees are affected by this negative shock to lagged fertility is analysed in this section. The aggregate impact on economic variables can be

[9] This procedure is standard in the RBC literature.

[10] It is usually more natural to think of a change in the retirement age as either endogenous to the agent or, alternatively, as exogenous and under government control. In this section, we are interested in the effects of a change in the retirement age and not in what causes that change.

[11] Formally, we restrict the model to consist of variables in the vector of endogenous variabels $\widehat{e}_t \in \{\widehat{k}_t, \widehat{c}_{1t}, \widehat{c}_{2t}, \widehat{l}_t, \widehat{y}_t, \widehat{R}_t, \widehat{w}_t, \widehat{\theta}_t\}$, as well as a reduced vector of exogenous state variables $\widehat{z}_t \in \{\widehat{\chi}_{t-1}, \widehat{\chi}_t, \widehat{b}_{t-1}, \widehat{\phi}_t^u, \widehat{\phi}_t^e\}$. The reduced model is therefore re-stated in terms of fewer state variables. The expression e.g. for leisure changes from (11) to: $\widehat{l}_t = \pi_{lk}\widehat{k}_{t-1} + \pi_{lc2}\widehat{c}_{2t-1} + \pi_{lb1}\widehat{b}_{t-1} + \pi_{l\phi u}\widehat{\phi}_t^u + \pi_{l\phi e}\widehat{\phi}_t^e + \pi_{l\chi 1}\widehat{\chi}_{t-1} + \pi_{l\chi}\widehat{\chi}_t$. The three remaining demographic changes $\{\widehat{b}_t, \widehat{a}_t, \widehat{\phi}_{t-1}^e\}$ could be analysed too, but this is beyond the scope of this paper.

decomposed into their subeffects by inspecting the relevant elasticities, see table 1. The results are presented both as analytical expressions, so their components can be interpreted, and as numerical simulations, in order to gain an understanding of the magnitudes involved. The interpretation of e.g. π_{lb1} is that a 1% increase in fertility will produce a 0.001% decrease in leisure. This will magnify the fertility-effect on the shrinking effective labour force and the increasing capital-labour ratio. This intuition behind this result warrants a more extensive scrutiny which we provide in the following.

TABLE 1. A SHOCK TO LAGGED FERTILTY

Effect on	*Analytical elasticity*	*Value*
$\pi_{c1b1} =$	$[\pi_{c2k} - \pi_{Rk}]\,\pi_{kb1}$	$= -0.01$
$\pi_{lb1} =$	$\pi_{c1b1} + \Lambda_{23}\pi_{\theta b1} - \Lambda_{22}\pi_{wb1}$	$= -0.001$
$\pi_{c2b1} =$	$\frac{\Lambda_{15}\pi_{lb1} - \Lambda_3\pi_{c1b1} - \Lambda_5\pi_{kb1} - \Lambda_2}{\Lambda_4}$	$= 0.34$
$\pi_{kb1} =$	$\frac{\Lambda_{12}\pi_{wb1} - \Lambda_{21}\pi_{lb1} - \Lambda_8\pi_{\theta b1}}{\Lambda_9\pi_{wk} - \Lambda_7\pi_{c2k} + \Lambda_{12}\pi_{Rk} - \Lambda_{20}\pi_{lk}}$	$= -0.01$

If intensive labour supply was exogenous, the only effect on the capital-labour ratio originates from the lower growth rate of the labour force, so the effects on consumption can be directly determined by wages and pension contributions, as in Jensen and Jørgensen (2008). On the other hand, if labour supply is a choice-variable both consumption and leisure are interrelated and together determine the capital-labour ratio. Consequently, we analyse the substitution, income, and wealth effects on leisure. This analysis will be founded on the IBC in (5).

For a 1% shock to lagged fertility, $\widehat{b}_{t-1}$, we insert the law of motion for the relevant variables into the log-linearised equation for leisure (12) in order to obtain the elasticity for leisure, $\widehat{l}_t$, with respect to $\widehat{b}_{t-1}$ in (13):

$$\widehat{l}_t = \widehat{c}_{1t} - \Lambda_{22}\widehat{w}_t + \Lambda_{23}\widehat{\theta}_t - \Lambda_{22}\pi_{\rho_1(b)}\widehat{b}_t \tag{12}$$

$$\pi_{lb1} = \pi_{c1b1} - \Lambda_{22}\pi_{wb1} + \Lambda_{23}\pi_{\theta b1} \tag{13}$$

Analysing first the *substitution effect* on $\widehat{l}_t$ we know that a smaller labour force generates higher wages and a lower interest rate. A higher wage rate will increase the price of $\widehat{l}_t$, so its substitution effect will be negative. The substitution effect on both first period consumption, $\widehat{c}_{1t}$, and second period consumption, $\widehat{c}_{2t}$, will be positive because their prices become relatively lower than that of $\widehat{l}_t$. An offsetting effect on the positive response of $\widehat{c}_{2t}$ is that the negative response of the interest rate will make the discounted price on $\widehat{c}_{2t}$ higher. The elasticity for second period consumption is $\pi_{c2b1} = 0.34$, which is then -0.34 for a negative shock to $\widehat{b}_{t-1}$. Since the prices on $\widehat{l}_t$ and $\widehat{c}_{2t}$ have increased, an unchanged level of income can buy less, so the income effects on both $\widehat{c}_{1t}$ and $\widehat{c}_{2t}$, as well as on $\widehat{l}_t$, are negative.

The increasing wage rate also appears in lifetime income on the right-hand side of IBC. This wealth effect is consequently positive for both $\widehat{c}_{1t}$, $\widehat{c}_{2t}$ and $\widehat{l}_t$. The dynamics of $\widehat{l}_t$ and $\widehat{c}_{2t}$, for a positive shock to $\widehat{b}_{t-1}$, are illustrated by the simulated trajectories in figures 2 and 3, respectively[12].

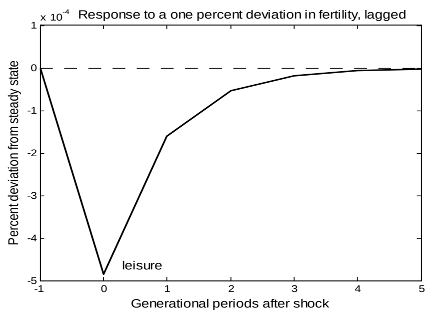

FIGURE 2. WORKERS' LEISURE

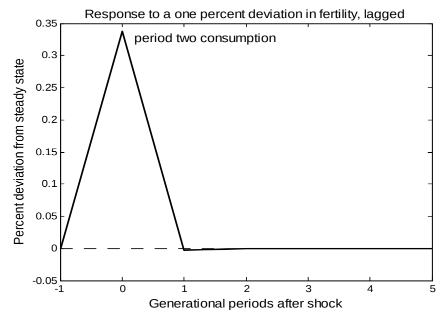

FIGURE 3. RETIREES' CONSUMPTION

If there is no distortionary taxation, then in our case, with an intertemporal elasticity of substitution equal to 1, these three effects will offset each other so the net effect on $\widehat{l}_t$ is zero. However, there is in fact a proportional contribution rate from the pension system associated with the price on $\widehat{l}_t$ and with lifetime income. The fact that proportional taxes are distorting when $\widehat{l}_t$ is endogenous means that the positive wealth effect will more than offset the negative sum of the substitution and income effects[13]. In this context Weil (2006) finds that the the most important means by which ageing will affect aggregate output and welfare is the distortion from taxes to fund PAYG pension systems. Thus, the elasticity for $\widehat{l}_t$ with respect to $\widehat{b}_{t-1} = -1$ is positive, i.e. $\pi_{lb1} = 0.001$.

Since workers earn wages and pay pension constributions we have to consider the "factor price effects" and the "fiscal effects" on $\widehat{c}_{1t}$, respectively. Regarding the fiscal effect, the negative fertility shock requires each worker to pay more taxes in order to finance the fixed benefits to retirees. Through the factor price effect, workers receive higher wages because of the higher capital-labour ratio caused by a direct effect and an indirect effect: the negative shock to fertility *directly* affects the population growth rate $(1+n_t)$. The *indirect* effect originates from the endogenous response of leisure $(\pi_{lb1} > 0)$. Consequently, the intensive labour supply $(-\pi_{lb1})$ falls, and the net effect on the capital-labour ratio is thus a net increase. The net effect on $\widehat{c}_{1t}$ is therefore ambiguous, but we can state the necessary condition under which the factor price effect will dominate the fiscal effect. By log-linearising c_{1t}

[12] The dynamics of $\widehat{c}_{1t}$ is identical to the simulated trajectory for $\widehat{l}_t$ in figure 2, though larger numerically. In figure 3, we see that $\widehat{c}_{2t}$ is negative in the next period $(t+1)$. This is because there will be effects on $\widehat{l}_t$ for a number of periods after the shock, which will still be (decreasingly) higher than the steady state value in the coming periods. This increases the capital-labour ratio and reduces interest rates. The lower interest rate in period $t+1$ therefore produces a negative change for current workers' retirement consumption, $\widehat{c}_{2t+1}$.

[13] If the contribution rate were equal to zero, $\theta = 0$, then $\pi_{\theta b1} = 0$, and the sum of the three effects would be zero so that $\pi_{lb1} = 0$. The distorting effects of taxation increase with the size of the pension PAYG system, so the larger θ is the larger is the difference between the positive wealth effect and the negative sum of substitution and income effects, and the more π_{lb1} increases numerically.

around steady state, we obtain $\widehat{c}_{1t} = \widehat{w}_t - (\frac{\theta}{1-\theta})\widehat{\theta}_t - (\frac{l}{1-l})\widehat{l}_t - \widehat{\chi}_t$. The expression is restated by inserting its law of motion:

$$\pi_{c1b1} = -\left(\alpha - \frac{\theta}{1-\theta}\right) + \left(\alpha - \frac{l}{1-l}\right)\pi_{lb1} \tag{14}$$

If labour supply is exogenous the expression reduces to the first bracket of (14), where the factor price effect is captured by α, and the fiscal effect by $\theta/(1-\theta)$, which is the only result in Bohn (2001). However, when labour supply is endogenous the factor price effect is adjusted in the second bracket of (14). This latter term indicates the adjustment of both the factor price effect and the fiscal effect by changes in labour supply. This implies that, given reasonable parameters, it will require either a very large pension system or an extremely high value on leisure in utility, to overturn the result that the factor price effect dominates the fiscal effect, even when labour supply is endogenous[14].

In a DC PAYG system pension contributions are fixed so the impact on $\widehat{c}_{1t}$ omits the fiscal effects, and only the positive factor price effect remains[15]. Consequently, workers would gain from switching to a DC system. The unchanging contributions to pensions yield a lower total amount of benefits to be distributed among retirees - thus replacements will fall and so will the rate of return on their savings. It is clear, therefore, that a DB system automatically transfers welfare-burdens, in terms of consumption and leisure, across generations.

3.2 A shock to life expectancy

An increase in life expectancy, $\widehat{\phi}^e_t$, is assumed to fall entirely on the length of the retirement period, so current workers will anticipate a longer retirement period which needs to be financed through higher savings. If labour supply was exogenous there would be no factor price effects because the capital-labour ratio would remain constant (see Jensen and Jørgensen, 2008, and Bohn, 2001). However, if labour supply is in fact endogenous a positive shock to $\widehat{\phi}^e_t$ affects the choice of $\widehat{l}_t$ and thus the capital labour ratio.

To derive the impact on economic variables of a 1% shock to $\widehat{\phi}^e_t$ we insert the law of motion for relevant variables in (12) and obtain the elasticities in table 2. In our numerical example the price on $\widehat{l}_t$ decreases ($\pi_{w\phi e} = -0.02$), and since the price on $\widehat{c}_{2t}$ has also decreased, an unchanged level of income can buy more, so the income effect on $\widehat{l}_t$, $\widehat{c}_{1t}$ and $\widehat{c}_{2t}$ is positive. However, lifetime income will decrease so the wealth effect is negative. We find that the negative wealth effect will more than offset the positive sum of the substitution and income effects, hence the net effect on $\widehat{l}_t$ is $\pi_{l\phi e} = -0.05$. This decrease in $\widehat{l}_t$ corresponds to an increase in the intensity of labour supply: $-\pi_{l\phi e} = 0.05$.

[14] The value of θ is derived through calibration for $\beta = 0.3$ such that $\frac{\theta}{1-\theta} = 0.28$. With a value of $\alpha = 1/3$, the pension system must be relatively large (with contribution rates above 30%), or labor supply must endogenously increase a lot, before this result is overturned. Weil (2006) finds a contribution rate in the US to be approx. 16% and increasing to 21% in 2030, if no government action is taken. Furthermore, the term $\frac{l}{1-l} = 0.67$.

[15] The difference between DB and DC systems is reflected by the elasticity for the response of the contribution rate to different shocks: in the DB system, $\pi_{\theta b1} = -1$, and in the pure DC system $\pi_{\theta b1} = 0$. As such, a DC system can easily be analysed by reversing the signs in the the log-linearised PAYG-equation: $\widehat{\theta}_t = \widehat{\phi}^e_{t-1} + \widehat{\phi}^u_t + \widehat{\mu}^e_{2t-1} + \widehat{\mu}^u_{2t} - \widehat{\chi}_{t-1} - \widehat{\chi}_t - \widehat{\mu}^e_{1t-1} - \widehat{\mu}^u_{1t} - \widehat{b}_{t-1}$.

Table 2. A shock to life expectancy

Effect on	*Analytical elasticity*	*Value*
$\pi_{c1\phi e} =$	$\left[\pi_{c2k} - \pi_{Rk}\right]\pi_{k\phi e} + \left(\pi_{c2\phi e1} - \pi_{R\phi e1}\right)$	$= -0.07$
$\pi_{l\phi e} =$	$\pi_{c1\phi e} + \Lambda_{23}\pi_{\theta\phi e} - \Lambda_{22}\pi_{w\phi e}$	$= -0.05$
$\pi_{c2\phi e} =$	$-\Lambda_{21}\pi_{l\phi e}$	$= \ 0.002$
$\pi_{k\phi e} =$	$\frac{\Lambda_{15}\pi_{l\phi e} - \Lambda_3\pi_{c1\phi e} - \Lambda_4\pi_{c2\phi e}}{\Lambda_5}$	$= \ 0.60$

The factor price effect, $\alpha\pi_{l\phi e}$, is negative due to the impact on wages of a higher labour supply. By working *more* intensively the contributions to the PAYG system is spread *less* intensively across the working period, hence the fiscal effect is positive on $\widehat{c}_{1t}$. The net effect on $\widehat{c}_{1t}$ is therefore, in principle, ambiguous but our numerical example shows that the fiscal effect will not counteract the fall in wages and the increase in savings ($\pi_{k\phi e} = 0.60$) so the net effect on $\widehat{c}_{1t}$ is negative.

Due to fixed benefits in the PAYG system the only effect on $\widehat{c}_{2t}$ comes from a positive indirect effect on the interest rate ($\pi_{R\phi e} = 0.04$) through the endogenous increase in the intensity of labour supply. As a result, $\widehat{c}_{2t}$ increases slightly ($\pi_{c2\phi e} = 0.002$). If labour supply was exogenous there would be no effect from the shock on neither $\widehat{l}_t$, the capital-labour ratio, nor the interest rate. This is the case in both Jensen and Jørgensen (2008) and Bohn (2001), where $\widehat{c}_{2t}$ is completely unaffected by changes in life expectancy[16]. Their result is now overturned, and we find that a higher preference for leisure in utility (γ) will put upward pressure on savings, the interest rate, and $\widehat{c}_{2t}$, as well as downward pressure on $\widehat{c}_{1t}$ and $\widehat{l}_t$. Further robustness analyses are provided below, when we consider the optimal policy responses to the shocks to fertility and life expectancy. Then clear message from these robustness analyses is that all results become numerically larger[17].

3.3 A shock to the retirement age

There will be economic effects of changes in the statutory retirement age. These effects should be well understood by policy makers, and the purpose of this section is to derive and interpret the impacts on key macroeconomic variables when the retirement age changes.

The analysis assumes an exogenous shock to the retirement age, without any presumption of who or what caused the change. This approach is chosen to emphazise only the economic effects of the change, and leave out any judgement of why the change has occured. Furthermore, a statutory retirement age is in fact exogenous to the economic decisions of households. What is endogenous to the household, on the other hand, is the *effective* retirement age, which in this paper is incorporated through endogenous labour supply. If the statutory retirement age

[16] If the pension contribution rate, θ, is zero the response of θ with respect to a shock to $\widehat{\phi}_t^e$ will be zero ($\pi_{\theta\phi e} = 0$). In that case, the net effect on leisure would also be zero ($\pi_{l\phi e} = 0$), and the distortionary effects of taxation increase numerically with the size of the pension system, and the effect on leisure will be increasingly negative.

[17] More details on the robustness of results can be obtained from the technical appendix, which is available upon request.

increases, households can decide to supply less labour – effectively reducing labour supply. This reduction could be assumed to take place in the end of the working period and, as such, reflect endogenous changes in the effective retirement age. In section 4, on optimal policy responses to demographic shocks, we employ the retirement age as a policy instrument, and this analysis takes into account the economic effects, derived in this section.

An increase in the retirement age is found to directly increase labour supply, and lower the length of the retirement period. As a result, workers save less. The effects on $\widehat{l}_t$ are determined directly by two elements: changes in the capital-labour ratio, and changes in pension contributions. When labour supply is endogenous the positive change in $\widehat{\chi}_t$ could *indirectly* affect $\widehat{l}_t$ and thus reinforce or reduce the *direct* effect on the capital-labour ratio. This is a case where an increase in the statutory retirement age reduces the intensity of labour supply. The effective labour supply consequently rises initially, because the retirement age has increased, but the fall in the intensive labour supply then reduces the effective labour supply. Our numerical simulation shows that $\widehat{l}_t$ is positive at $\pi_{l\chi} = 0.09$ (see table 3), so the net impact on effective labour supply is a 0.89% increase, because the statutory retirement age increases by 1%, so the endogenous response of the intensity of labour supply is a fall by 0.09%.

Table 3. A shock to the retirement age

Effect on	*Analytical elasticity*	*Value*
$\pi_{c1\chi} =$	$[\pi_{c2k} - \pi_{Rk}]\,\pi_{k\chi} + (\pi_{c2\chi1} - \pi_{R\chi1})$	$= 0.12$
$\pi_{l\chi} =$	$\frac{[\pi_{c2k}-\pi_{Rk}]\pi_{k\chi}+(\pi_{c2\chi1}-\pi_{R\chi1})+(\Lambda_{23}\pi_{\theta\chi}+\Lambda_{22}\Lambda_{11})}{1+\Lambda_{22}\Lambda_{11}}$	$= 0.09$
$\pi_{c2\chi} =$	$[\Lambda_9\pi_{wk} - \Lambda_7\pi_{c2k} + \Lambda_{12}\pi_{Rk} - \Lambda_{20}\pi_{lk}]\,\pi_{k\chi}$ $-\Lambda_{21}\pi_{l\chi} + \Lambda_{12}\pi_{w\chi} - \Lambda_8\pi_{\theta\chi}$	$= 0.23$
$\pi_{k\chi} =$	$\frac{\Lambda_{15}\pi_{l\chi}-\Lambda_3\pi_{c1\chi}-\Lambda_4\pi_{c2\chi}-\Lambda_2}{\Lambda_5}$	$= -0.55$

The substitution effect on $\widehat{l}_t$ is positive because the the price on $\widehat{l}_t$ decreases ($\pi_{w\chi} = -0.30$). Return to savings increases by $\pi_{R\chi} = (1-\alpha)(1-\pi_{l\chi}) = 0.60$, where the *direct* effect from $\widehat{\chi}_t$ is $(1-\alpha)$ due to the lower capital-labour ratio, but since labour supply is endogenous this effect is *indirectly* reduced by $(1-\pi_{l\chi})$. The net effect on the capital-labour ratio is still negative, and thus the net effect on returns remains positive. Consequently, the price on $\widehat{c}_{2t}$ falls, along with the price on $\widehat{l}_t$, and retirees gain by $\pi_{c2\chi} = 0.23$. Furthermore, lifetime income falls because the wage rate falls (factor price effect). In our case, this negative wealth effect will not offset the positive sum of the substitution and income effects. Thus, $\pi_{l\chi} = 0.09$ is positive.

In terms of robustness analysis, figure 4 illustrates that a higher preference for leisure will increase the elasticity for $\widehat{l}_t$ with respect to $\widehat{\chi}_t$. This implies that if households over time weigh (demand) leisure to an increasing extent, e.g. in line with economic prosperity, then an increase in the retirement age, which yields less lifetime leisure, will induce them to supply less and less labour.

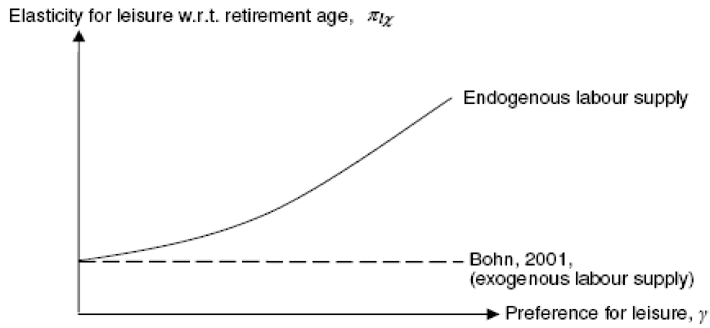

FIGURE 4. LEISURE AND THE RETIREMENT AGE

Workers now have more subperiods during which they can contribute to the fixed benefits of retirees. As they save less they free resources for $\widehat{l}_t$ and $\widehat{c}_{1t}$, and this leads to a positive fiscal effect. The factor price effect is negative, so lower wages must now be spread over a longer working period. The net effect on $\widehat{c}_{1t}$ is therefore theoretically ambiguous, but our simulations show that $\pi_{c1\chi} = 0.12$ is positive, which leads us to the conclusion that the negative response of savings, together with the positive fiscal effect, is large enough to outweigh the negative factor price effect, the negative effect on labour supply, and the increased financing of a longer working period.

The combination of a negative shock to $\widehat{b}_{t-1}$ and a positive shock to $\widehat{\phi}_t^e$ embodies identical mechanisms as a positive change in $\widehat{\chi}_t$. This is because both a positive change in $\widehat{b}_{t-1}$ and in $\widehat{\chi}_t$ increases the effective labour force. In addition, a positive shock to $\widehat{\phi}_t^e$ increases current workers' retirement period. Similarly, when the $\widehat{\chi}_t$ increases current workers expect a shorter retirement period.

4 Policy Reform

By analysing the effects of shocks to fertility and life expectancy we have seen that three main forces are operating: the *endogenous intensity of labour supply*, the *factor price effect*; and the *fiscal effect*. The fiscal effect originates from the passive pension contribution rate, which plays a major role for how the welfare effects are distributed across generations. In general, we found that the fiscal effects were not sufficient to counteract the net factor price effects and, consequently, workers and retirees were exposed unevenly to the changes in fertility and life expectancy. For that reason we now consider a more *active* policy rule for the retirement age, in order to achieve more socially desirable outcomes. In order to evaluate the social desirability of the results obtained in section 3, we compare those results to a socially optimal allocation which is in accordance with society's preferences. If this optimal allocation differs from the market allocation, we need to consider redistributional policies.

4.1 Welfare

This section assumes a welfare function in (15) to be maximised by the social planner subject to the resource constraint in (8) and the utility function in (2),

$$\Upsilon = E\left\{\sum_{t=-1}^{\infty}\Phi_t N_t^w U_t\right\} \equiv \left\{\Phi_{t-1}N_{t-1}^w U_{t-1} + \Phi_t N_t^w U_t\right\} \tag{15}$$

where $\Phi's$ are weights on each generation's utility in the welfare function. Assuming that socity's preferences are very egalitarian, equal assigned weights to the utility of each generation, i.e. $\Phi_{t-1} \equiv \Phi_t$.

The social planner derives two optimality conditions by equating the following derivatives: $\partial\Upsilon_t/\partial c_{1t} = \Phi_t N_t^w \partial U_t/\partial c_{1t}$, $\partial\Upsilon_t/\partial l_t = \Phi_t N_t^w \partial U_t/\partial l_t$, and $\partial\Upsilon_t/\partial c_{2t} = \Phi_{t-1}N_{t-1}^w \partial U_{t-1}/\partial c_{2t}$. We derive the intergenerational optimality condition in (16) and the consumption-leisure optimality condition in (17), both stated in efficiency units and log-linearised around steady state[18].

$$\widehat{c}_{1t} = \widehat{c}_{2t} + \pi_{\rho 1(b)}\widehat{b}_t \tag{16}$$

$$\widehat{c}_{1t} = \widehat{l}_t + \pi_{\rho 1(b)}\widehat{b}_t - \widehat{\chi}_t \tag{17}$$

Combining (16) and (17) yields the social planner's welfare optimality condition in (18) that ensures equal responses for workers and retirees to demographic changes, which is the socially desirable outcome[19].

$$\widehat{c}_{2t} = \widehat{l}_t - \widehat{\chi}_t \tag{18}$$

This condition for optimal intergenerational risk sharing reveals that the percentage change in workers' *lifetime* leisure ($\widehat{l}_t$ adjusted by $\widehat{\chi}_t$ because of the disutility of less lifetime leisure) should equal the percentage change in the consumption of retirees[20]. In case this is not replicated by the market equilibrium, economic policy should modify this outcome by redistributing intergenerationally up to the point where all generations are affected in equal proportions. In this paper we argue that this could be achieved by changing in the retirement age.

[18] When welfare is maximised, the problem is defined over just two generations: current workers and current retirees. Current workers will be retirees in the next period, and at that time their utility is weighted relative to the utility of those who are currently children. In this way the welfare of workers are always maximised relative to the welfare of retirees. The welfare optimality conditions in (16) and (17) therefore hold for all future periods and not just for workers and retirees in the *present* period. For instance, it is clear from (16) that in each period in the future the consumption of any generation of workers should always respond in the same proportions as the consumption of any generation of retirees, e.g. in the next period $\widehat{c}_{2t+i} = \widehat{l}_{t+i} - \widehat{\chi}_{t+i}$. A given redistribution of income, which leaves $\widehat{c}_2$ equal to $\widehat{l}$ in the present perio, will also leave $\widehat{c}_2$ equal to $\widehat{l}$ in all future periods. In that way, the impacts of a demographic shock is shared by *current* as well as *future* generations.

[19] The log-linearised factor $\rho_{1(\widehat{b}_t)}$ is defined as $\pi_{\rho 1(b)}\widehat{b}_t$, where $\pi_{\rho_{1(b)}}$ is elasticity of the weight on first period consumption in utility (calibrated equal to 1 in the numerical analysis).

[20] Bohn (2001) defines this as *efficient* risk sharing.

4.2 Optimal policy response

This section applies the welfare optimality condition in (18) in combination with the laws of motion for $\widehat{c}_{2t}$ and $\widehat{l}_t$, in order to obtain the set of optimal allocations. Subsequently we solve for the optimal policy response of the retirement age for two purposes: *first*, in order to offset the decline in the labour force; and *secondly*, in order to achieve optimal intergenerational risk sharing.

4.2.1 The labour force

In this section, we make use of our general equilibrium framework to derive how much the retirement age should increase in order to offset the decline in the labour force historically caused by low fertility. It is important, though, which role one assigns to the variable $\widehat{\chi}_t$, and we adopt the approach of treating $\widehat{\chi}_t$ as an exogenous variable that is under government control. The result is independent of the social desirability of any intergenerational distribution of the associated effects. The *effective* labour supply is $d_t = (1 - l_t)(1 + n_t^w)$, or in log-deviations from it's steady state value[21]:

$$\widehat{d}_t = \widehat{\chi}_t + \widehat{b}_{t-1} - \widehat{l}_t \tag{19}$$

Assume first that the intensity of labour supply is *exogenous* and that we examine a 1% decline in fertility. It is then clear from (19) that the necessary response of the retirement age, which would offset the fertility decline, i.e. $\widehat{d}_t = \widehat{\chi}_t + \widehat{b}_{t-1} - \widehat{l}_t \equiv 0$, would just be a proportional increase of $\widehat{\chi}_t = 1\%$. However, if the intensity of labour supply is indeed endogenous, so $\widehat{l}_t \neq 0$, of course the response of the retirement age would have to be different from 1%. In our case, where leisure increases, the initial effect from the fertility decline on the effective labour supply will be *reinforced*, and the retirement age would have to increase even *more* than 1%. To derive the optimal response of $\widehat{\chi}_t$ we insert the linear law of motion for $\widehat{l}_t$,

$$\widehat{\chi}_t = [\pi_{lb1}\widehat{b}_{t-1} + \pi_{l\chi}\widehat{\chi}_t] - \widehat{b}_{t-1}$$

then isolate $\widehat{\chi}_t$, and insert the numerical elasticities for $\widehat{b}_{t-1} = -1$:

$$\widehat{\chi}_t = -\left[\frac{1 - \pi_{lb1}}{1 - \pi_{l\chi}}\right]\widehat{b}_{t-1} = 1.10 \tag{20}$$

Observe that if $\pi_{lb1} < \pi_{l\chi}$ the optimal response is $\widehat{\chi}_t > 1$, so we conclude that the retirement age has to increase *more* than fertility *fell* in order to offset the negative impact on the effective labour force. This is due to the choice of leisure by individuals, which will increase both when fertility falls and when the retirement age increases and thus further lower labour supply. The offsetting response of the retirement age, when $\widehat{b}_{t-1} = -1\%$, is derived to be an increase of $\widehat{\chi}_t = 1.10\%$.

In terms of robustness analysis, when the relative weight on leisure in utility, γ, increases so will the optimal response of the retirement age. This is intuitive because the responses of leisure (π_{lb1} and $\pi_{l\chi}$) will be larger in size and in discrepancy. Consequently, there is a tendency for labour supply to fall even further, and this must be counteracted by larger and larger increases in the retirement age.

[21] In terms of our analytical framework, the growth rate of the population is $n_t^w = 1 + b_{t-1}$, and the *net* growth rate is $n_t^w = \chi_t b_{t-1}$. We have to incorporate the intensity of labour supply as well, though, and this is accounted for by scaling net labour supply by $(1 - l_t)$.

4.2.2 The retirement age

Proposals for analysing the retirement age as a policy instrument are found in e.g. de la Croix et al (2004) and Andersen, Jensen and Pedersen (2004)[22]. By considering the composite shock to fertility of $\widehat{b}_{t-1} = -1\%$ and life expectancy $\widehat{\phi}_t^e = 1\%$ the recursive equilibrium law of motion for $\widehat{l}_t$ and $\widehat{c}_{2t}$ can be inserted into (18) to yield (21). Solving for $\widehat{\chi}_t$ provides the percentage change in the retirement age that ensures a socially optimal intergenerational allocation in (22), where the optimal policy response is $\widehat{\chi}_{t|\widehat{b}_1,\widehat{\phi}^e} = 0.25\%$.

$$\pi_{c2\chi}\widehat{\chi}_t + \pi_{c2b1}\widehat{b}_{t-1} + \pi_{c2\phi e}\widehat{\phi}_t^e = \pi_{l\chi}\widehat{\chi}_t + \pi_{lb1}\widehat{b}_{t-1} + \pi_{l\phi e}\widehat{\phi}_t^e - \widehat{\chi}_t \tag{21}$$

$$\widehat{\chi}_{t|\widehat{b}_1,\widehat{\phi}^e} = \left[\frac{\pi_{lb1} - \pi_{c2b1}}{\pi_{c2\chi} - (\pi_{l\chi} - 1)}\right]\widehat{b}_{t-1} + \left[\frac{\pi_{l\phi e} - \pi_{c2\phi e}}{\pi_{c2\chi} - (\pi_{l\chi} - 1)}\right]\widehat{\phi}_t^e = 0.25 \tag{22}$$

The composite shock entails dynamics that turns out to be counteracted by the change in the retirement age, and the sum of effects for $\widehat{c}_{2t}$ and workers' *lifetime* leisure $(\widehat{l}_t - \widehat{\chi}_t)$ is exactly the same when the policy rule for the retirement age is $\widehat{\chi}_t = 0.25\%$. By disaggregating the dynamics we find that the optimal reponse of $\widehat{\chi}_t$ for a shock to *only* $\widehat{b}_{t-1}$ is $\widehat{\chi}_t = 0.30$, and for a shock to *only* $\widehat{\phi}_t^e$ is $\widehat{\chi}_t = -0.05$. Adding these partial results, yields the net result of exactly $\widehat{\chi}_t = 0.25\%$. There are two key dynamics in play: *First*, a negative shock to $\widehat{b}_{t-1}$ will reduce the current labour force and bring about a series of effects on other macroeconomic variables. A policy response of $\widehat{\chi}_t > 0$ will counteract this reduction in the labour force, as well as all the initial effects on other variables. *Secondly*, the retirement *period* is residually lowered by an increase in the retirement *age*, but the increase in $\widehat{\phi}_t^e$ counteracts this effect[23].

In terms of robustness analysis, if the weight on leisure in utility (γ) *increases* we obtain a *smaller* optimal response of χ. This seems plausible, since a higher γ causes higher disutility of less lifetime leisure for workers. Consequently, the more valued leisure is the less the retirement age has to increases to equalise the welfare responses of workers and retirees. The policy recommendation is therefore to establish an *indexation scheme of retirement age relative to life expectancy* in order to ensure optimal welfare distributions across generations[24].

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[22] In addition, Cutler (2001) recommends an extention of Bohn (2001) to incorporate "the length of the period where people work".

[23] Jensen and Jørgensen (2007) obtain a 1:1 optimal response of the retirement age because, first, they do not scale leisure in utility by the retirement age, and second, they do not have endogenous labour supply in their model.

[24] Even though we argue that an increase in taxes may not be realistic and optimal in relation to labour supply incentives and distortions we can still derive the optimal response of the contribution rate for the composite shock to fertility and life expectancy. We derive that the optimal response of the contribution rate should equal: $\pi_{\theta|\widehat{b}_1,\widehat{\phi}^e} = \frac{[\pi_{l\phi e} - \pi_{c2\phi e}]\omega_6\omega_{12} - [\pi_{c2k} - \pi_{Rk}]\omega_7 + \omega_{11} - \omega_9}{\omega_{10}\omega_{12} - \Lambda_{23}\omega_6 + \omega_2\omega_8} = -1.9$, which means that for $\widehat{b}_{t-1} = -1$ the contribution rate should increase by approximately 2%, This is more than the 1% it automatically does through the *passive* PAYG system. For a detailed account of this analysis consult Jensen and Jørgensen (2007). Alternatively, a detailed appendix to this paper is available upon request (ω's denote combinations of steady state variables and model parameters).

5 Conclusion

Fertility decline and increases in life expectancy excert strong upward pressure on dependency ratios. A key impact on welfare is that different generations may be affected in different proportions, which may not be socially desirable, so there may be a role for government policy to redistribute across generations to achieve a more equitable outcome.

The retirement age is likely to be the preferred policy instrument, so we incorporate the retirement age into a stochastic OLG model with endogenous labour supply and ask two questions: first, what are the effects on different generations in terms of welfare? Secondly, can a policy rule for the retirement age be designed in order to offset any adverse dynamics? We find that workers and retirees are affected unevenly for changes in fertility, life expectancy, and the retirement age, respectively. The endogeneity of labour supply is found to either counteract or reinforce these changes - which has a major impact on the distribution of welfare across generations.

We have then considered whether these unequal distributions are in accordance with social preferences. This is not the case, and redistribution is therefore necessary in order to equalise the effects across generations. With this motivation we derive an optimal policy rule for the retirement age.

We are interested in adapting this policy rule for the retirement age to two circumstances: first, by how much should the retirement age increase to offset the decline in the labour force due to the historical fertility fall? We find that the retirement age has to increase by more than fertility fell, i.e. the optimal increase in the retirement age should be 1.1% whenever fertility decreases by 1%. This is because labour supply endogenously falls when fertility falls; when life expectancy increases; and when the retirement age increases.

Second, by how much should the retirement age change to redistribute the welfare burdens more equally across generations in accordance with social preferences? In this case an optimal policy rule is also to increase the retirement age, taking into account, however, the disutility for workers of less lifetime leisure when the retirement age increases. For that reason, the optimal response of the retirement age is less than 1:1 with the change in life expectancy. A policy rule where the retirement age is linked to life expectancy in a 1:4 relationship is found to ensure that both workers and retirees bear the burdens of the demographic changes in proportions that are more equitable.

We find that leisure increases when the retirement age increases so this could be interpreted as an endogenous change in the voluntary early retirement age. In this perspective, we argue that increasing the retirement age will induce workers to retire earlier, based on their own financing, and this is exactly the opposite of what is intended by the policy rule. As a result the retirement age has to increase even more than it should when leisure is not an important welfare good. This argument is founded on our key finding that leisure tends to increase when fertility falls and to decrease when life expectancy increases.

Our analyses are very robust to changes in the weight on leisure in utility, which is a relatively unknown parameter. However, the analytical framework is subject to a couple of limitations. The utility function has been modelled in accordance with our best beliefs of how to incorporate the value of leisure and the lengths of periods. Our future research will examine the robustness of our result in more detail for varying specifications of the utility function. In addition, we assume that the economic impacts of changes in dependency ratios can be analysed in a linearised model. Simulation excercises with CGE models could in the future be performed for more empirical perspectives on our approach, but in this paper we highlight the analytical tractability that allows for an exact solution for policy rules. This is not feasible with a CGE model.

The fact that the valuntary early retirement age in developed countries has a negative correlation with life expectancy means that the latter should perhaps be endogenous to the former. A related question in the ageing debate is also whether the additional years that people are expected to live will actually be more or less healthy years. All of the above extensions are relevant to future research. The derivation of an optimal policy rule for the retirement age, in a setting with endogenous labour supply, is a productive starting point.

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A Solving Stochastic OLG Models Analytically[25]

In this section a closed form solution for the stochastic OLG model is derived. The method of undetermined coefficients (M.U.C.) is used to obtain an "analytical" solution for the recursive equilibrium law of motion - charaterized by providing the solution in terms of analytical elasticities[26]. By adopting this approach the non-linear OLG model is replaced by a log-linearised approximate model with variables in percentage deviations from the steady state. A less advanced version of the M.U.C. was first applied on OLG models by Andersen (1996, 2001) and Bohn (1998, 2001) in models without endogenous labour supply and without an indicator for length of working period.

When endogenous labour supply is incorporated in this paper, together with the pension system, the less advanced version of M.U.C. applied by Bohn (2001) and developed in Jørgensen (2006) can no longer be applied. These papers employ only one state variable, but now at least one additional state variable must be defined to solve the model, which is feasible with a version in matrix notation of M.U.C. in Uhlig (1999). This solution method has to our knowledge not previously been applied in the literature in the context of stochastic OLG models.

In our stochastic OLG model there are not only variables for *current* demographic changes but also for *expected future* demographic changes[27]. The M.U.C. in Uhlig (1999), which is stated in matrix notation, due to the ultimate solution of a generalized eigenvalue problem in matrix notation, cannot analyse expected future demographic changes in its current setup. We consequently label this a

[25] This appendix provides a short version of the analytical steps one must go through to solve this type of model with the method of undetermined coefficients. A version of this appendix that contains all derivations are available upon request.

[26] The term "analytical" is used in this context since the solution of the model relies not directly on numerical simulations, which is the usual practice for OLG models, but instead on an algebraic derivation of the model.

[27] For instance the length of life in $t+1$, denoted by ϕ_{t+1} in the model.

"Matrix-based" method and accept that it can be used only to solve for *current* demographic changes. However, a method that can actually handle the *expected future* demographic changes is subsequently developed in this paper, in which equations are kept in their original log-linearised form. Therefore, this is labeled an "Equation-based" method.

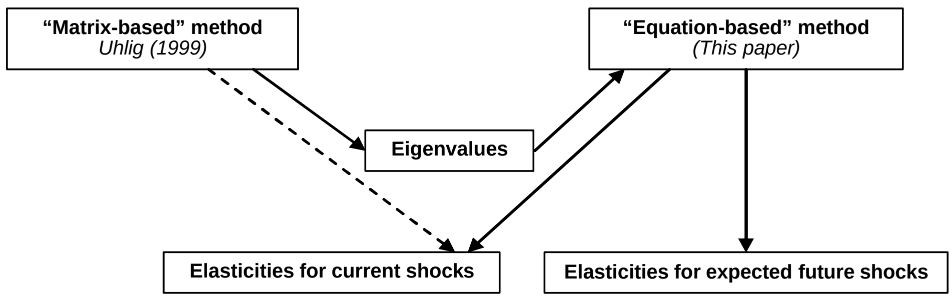

Figure 1A. The combination and output of methods

Our discussion above implies that if intensive labour supply is to be endogenized, and if the model is to be solved analytically, then a combination of the Matrix-based and the Equation-based methods must be applied in order to handle both *current* and *expected future* demographic changes. In this section a procedure for this combination is developed, and a representation of the important link between the methods is illustrated in Figure 1A. The Equation-based method builds on the Matrix-based method's ability to solve a *matrix quadratic equation* as a generalized eigenvalue problem and find the elasticities of endogenous state variables with respect to (w.r.t) their own lagged values (the eigenvalues). *Both* methods can be used to derive elasticities for current demographic changes, but only given that the Matrix-based method is first used to derive the eigenvalues. As such, only the *Matrix-based* method is capable of deriving the *eigenvalues*; and only the Equation-based method is capable of deriving the elasticities of *expected future* demographic changes.

Since we are interested in the *analytical* expressions for elasticities we only use the Matrix-based method to derive eigenvalues, and use instead the Equation-based method to derive all the expressions both for current and expected future demographic changes, respectively. In figure 2, therefore, the irrelevant arrow is punctured. The structure of the remainder of this section will be in accordance with the procedure we have developed for obtaining an analytical solution to the stochastic OLG model with the combination of a Matrix-based and an Equation-based M.U.C.:

1. *Solution of the model with the Matrix-based M.U.C. - with the purpose of deriving eigenvalues*

2. *Development and application of the Equation-based M.U.C. with the purpose of deriving elasticities for current and expected future demographic changes - based on the eigenvalues derived with the Matrix-based method*

A.1 Matrix-based Method of Undetermined Coefficients

The solution method for our stochastic OLG model is based on the analytical approach to solving stochastic dynamic general equilibrium models with the M.U.C. in Uhlig (1999)[28]. The key technical innovation in this paper is to combine a stochastic OLG model with an analytical solution method, which is constructed to handle an unlimited number of state variables with the ultimate solution of a generalized eigenvalue problem. Before we solve the model, following the procedure described below, the model is log-linearised around steady state, such that the original non-linear model is replaced by an approximate log-linearised model[29]. Variables denoted with "hats" are log-linearised variables in percent deviations from steady state, while variables without subscripts are steady state variables[30].

All endogenous variables from the log-linearised model, $\widehat{e}_t \in \{\widehat{k}_t, \widehat{c}_t^2, \widehat{c}_t^1, \widehat{l}_t, \widehat{y}_t, \widehat{R}_t, \widehat{w}_t, \widehat{\theta}_t\}$, are written as linear functions of a vector of endogenous and exogenous state variables, respectively. The vector of endogenous state variables is $\widehat{x}_t \in \{\widehat{k}_t, \widehat{c}_t^2\}$ of size $m \times 1$[31], the vector of endogenous non-state variables is $\widehat{v}_t \in \{\widehat{c}_t^1, \widehat{l}_t, \widehat{y}_t, \widehat{R}_t, \widehat{w}_t, \widehat{\theta}_t\}$ of size $j \times 1$, while the vector of exogenous state variables is $\widehat{z}_t \in \{\widehat{\chi}_{t-1}, \widehat{\chi}_t, \widehat{a}_t, \widehat{b}_{t-1}, \widehat{b}_t, \widehat{\phi}_{t-1}^e, \widehat{\phi}_t^e, \widehat{\phi}_t^u\}$ of size $g \times 1$. The log-linearised equations are in written matrix notation in the following equilibrium relationships,

$$0 = \mathbf{A}\widehat{x}_t + \mathbf{B}\widehat{x}_{t-1} + \mathbf{C}\widehat{v}_t + \mathbf{D}\widehat{z}_t \tag{23}$$

$$0 = E_t\left[\mathbf{F}\widehat{x}_{t+1} + \mathbf{G}\widehat{x}_t + \mathbf{H}\widehat{x}_{t-1} + \mathbf{J}\widehat{v}_{t+1} + \mathbf{K}\widehat{v}_t + \mathbf{L}\widehat{z}_{t+1} + \mathbf{M}\widehat{z}_t\right] \tag{24}$$

$$\widehat{z}_{t+1} = \mathbf{N}\widehat{z}_t + \varepsilon_{t+1}, \qquad E_t\left[\varepsilon_{t+1}\right] = 0 \tag{25}$$

where $\mathbf{C}$ is of size $h \times j$, where h denotes the number of non-expectational equations. In this particular OLG model $h = j$, due to the definition of $\widehat{x}_t = \{\widehat{k}_t, \widehat{c}_t^2\}$, because with merely the capital stock as a state variable $h < j$, and the system cannot not be solved[32]. The matrix $\mathbf{F}$ is of size $(m + j - h) \times j$, and it is assumed that $\mathbf{N}$ has only stable eigenvalues.

The recursive equilibrium is characterized by a conjectured linear law of motion between endogenous variables in the vector $\widehat{e}_t$, and state variables (endogenous and exogenous, respectively) in the vectors $\widehat{v}_t$ and $\widehat{z}_t$. The conjectured linear law of motion is written as,

$$\widehat{x}_t = \mathbf{P}\widehat{x}_{t-1} + \mathbf{Q}\widehat{z}_t \tag{26}$$

[28] The solution for eigenvalues in this section relies directly on a certain special case of the method in Uhlig (1999) to whom we refer for details of propositions and proofs. His method is inspired by the method in Campbell (1994).

[29] The rules for log-linearization are standard (see Uhlig, 1999). However, when growth rates are involved in the log-linearization process it is assumed that $(1 + a_t) \equiv a_t$ and in steady state $(1 + a) \equiv a$, i.e. the term "1+" surpressed for notational convenience, so that $\widehat{a}_t = \ln(a_t) - \ln(a)$ instead of $\widehat{a}_t = \ln(1 + a_t) - \ln(1 + a)$. The Λ's and ω's in the equations are coefficients composed by steady state variables.

[30] The equations that characterize the equilibrium of the stochastic OLG economy and must be log-linearized are: *the resource constraint*, *second period consumption*, *the Euler equation*, *the consumption-leisure optimality condition*, *income*, *wages*, *returns*, and *the PAYG system*.

[31] In order to solve the model it is necessary to have at least as many state variables as there are expectational equations in the model ($h \geq j$).

[32] Note that if $h > j$ the equations in this section become slightly more complicated, see Uhlig (1999), but a solution is still feasible.

$$\widehat{v}_t = \mathbf{R}\widehat{x}_{t-1} + \mathbf{S}\widehat{z}_t \tag{27}$$

where the coefficients in the matrices $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$ are interpreted as elasticities. These linear relationships between endogenous variables and state variables could alternatively be written out for each variable in $\widehat{e}_t$., as e.g for leisure, $\widehat{l}_t$,

$$\begin{aligned}\widehat{l}_t \;\; = \;\; & \pi_{lk}\widehat{k}_{t-1} + \pi_{lc2}\widehat{c}_{2t-1} \\ & + \pi_{l\chi 1}\widehat{\chi}_{t-1} + \pi_{l\chi}\widehat{\chi}_t + \pi_{la}\widehat{a}_t + \pi_{lb1}\widehat{b}_{t-1} + \pi_{lb}\widehat{b}_t + \pi_{l\phi e1}\widehat{\phi}^e_{t-1} + \pi_{l\phi e}\widehat{\phi}^e_t + \pi_{l\phi u}\widehat{\phi}^u_t\end{aligned}$$

where e.g. π_{la} denotes the elasticity (π) of leisure (l) w.r.t. productivity (a). The stability of the system is determined by the stability of the matrix $\mathbf{P}$, given the assumptions on the matrix $\mathbf{N}$. By inserting (26) and (27) into the non-expectational equation (23) the matrices $\mathbf{R}$ and $\mathbf{S}$ can be derived to be,

$$\mathbf{R} = -\mathbf{C}^{-1}\left(\mathbf{AP} + \mathbf{B}\right) \tag{28}$$

$$\mathbf{S} = -\mathbf{C}^{-1}\left(\mathbf{AQ} + \mathbf{D}\right) \tag{29}$$

The matrix-quadratic equation (30) to be solved for the matrix $\mathbf{P}$ and equation (31) to be solved for $\mathbf{Q}$ can be derived by inserting equations (25)-(28) into the expectational equation (24). From (30) the matrix-quadratic equation in (32) emerges, composed by (34)-(36). From (31) the matrix $\mathbf{Q}$ can be identified in (33) where $\mathbf{I}_g$ is the identity matrix of size $g \times g$, following Uhlig (1999).

$$0 = \left(\mathbf{F} - \mathbf{JC}^{-1}\mathbf{A}\right)\mathbf{P}^2 + \left(\mathbf{JC}^{-1}\mathbf{B} - \mathbf{G} + \mathbf{KC}^{-1}\mathbf{A}\right)\mathbf{P} - \left(\mathbf{KC}^{-1}\mathbf{B} - \mathbf{H}\right) \tag{30}$$

$$0 = \mathbf{FPQ} + \mathbf{FQN} + \mathbf{GQ} + \mathbf{JRQ} + \mathbf{JSN} + \mathbf{KS} + \mathbf{LN} + \mathbf{M} \tag{31}$$

$$0 = \mathbf{\Psi P}^2 + \mathbf{\Gamma P} - \mathbf{\Theta} \tag{32}$$

$$\begin{aligned} & vec\left[\left(\mathbf{JC}^{-1}\mathbf{D} - \mathbf{L}\right)\mathbf{N} + \mathbf{KC}^{-1}\mathbf{D} - \mathbf{M}\right] \\ = \;\; & \left[\mathbf{N}' \otimes \left(\mathbf{F} - \mathbf{JC}^{-1}\mathbf{A}\right) + \mathbf{I}_g \otimes \left(\mathbf{JR} + \mathbf{FP} + \mathbf{G} - \mathbf{KC}^{-1}\mathbf{A}\right)\right] vec\left(\mathbf{Q}\right)\end{aligned} \tag{33}$$

$$\mathbf{\Psi} = \mathbf{F} - \mathbf{JC}^{-1}\mathbf{A} \tag{34}$$

$$\mathbf{\Theta} = \mathbf{KC}^{-1}\mathbf{B} - \mathbf{H} \tag{35}$$

$$\mathbf{\Gamma} = \mathbf{JC}^{-1}\mathbf{B} - \mathbf{G} + \mathbf{KC}^{-1}\mathbf{A} \tag{36}$$

It is clear from equations (28)-(30) and (33) that in order to obtain a solution it is required that $\mathbf{C}$ is invertible[33]. The matrix-quadratic equation (32) is solved as a generalized eigenvalue-eigenvector problem, where the generalized eigenvalue, δ, and eigenvector, q, of matrix $\mathbf{\Xi}$ with respect to $\mathbf{\Delta}$ are defined to satisfy,

$$\delta\mathbf{\Delta}q = \mathbf{\Xi}q$$

[33] In case $\mathbf{C}$ is not directly invertible, Uhlig (1999) provides the method for a pseudo-inverse.

$$0 = (\boldsymbol{\Xi} - \delta\boldsymbol{\Delta})\, q$$

given that matrices $\boldsymbol{\Xi}$ and $\boldsymbol{\Delta}$ are defined as:

$$\boldsymbol{\Xi} = \begin{bmatrix} \boldsymbol{\Gamma} & \boldsymbol{\Theta} \\ \mathbf{I}_g & \mathbf{0}_{g,g} \end{bmatrix}$$

$$\boldsymbol{\Delta} = \begin{bmatrix} \boldsymbol{\Psi} & \mathbf{0}_{g,g} \\ \mathbf{0}_{g,g} & \mathbf{I}_g \end{bmatrix}$$

For this particular OLG model $\boldsymbol{\Delta}$ is invertible so the generalized eigenvalue problem can be reduced to a standard eigenvalue problem of solving instead the expression $\boldsymbol{\Delta}^{-1}\boldsymbol{\Xi}$ for eigenvalues-eigenvectors, as in (37). Then, $\boldsymbol{\Delta}^{-1}\boldsymbol{\Xi}$ is diagonalized in (38) since each eigenvalue, δ_i, can be associated with a given eigenvector, q_m.

$$\left(\boldsymbol{\Delta}^{-1}\boldsymbol{\Xi} - \delta\mathbf{I}\right) q = 0 \tag{37}$$

$$\mathbf{P} = \boldsymbol{\Omega}\boldsymbol{\Delta}^{-\mathbf{1}}\boldsymbol{\Xi}\boldsymbol{\Omega}^{-1} \tag{38}$$

The matrix $\boldsymbol{\Delta}^{-1}\boldsymbol{\Xi} = diag\,(\delta, ..., \delta_m)$ then contains the set of eigenvalues from which a saddle path stable eigenvalue can be identified, and the matrix $\boldsymbol{\Omega} = [q_1, ..., q_m]$ contains the characteristic vectors. Ultimately, the matrix $\mathbf{P}$, governing the dynamics of the OLG model, is derived, and the system can be "unfolded" to provide the elasticities in the matrices $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$.

The elasticities of endogenous variables with respect to *current demographic changes* have now been derived. The *expected future demographic changes* (exogenous state variables in period $t+1$) cannot be treated directly by this method, however, since the Matrix-based method, by construction, is only capable of handling demographic changes in period t. Therefore, the next section will develop the *Equation-based* method to analyse the elasticities of variables with respect to expected future demographic changes.

A.2 Equation-based Method of Undetermined Coefficients

The Equation-based method has a strong link to the Matrix-based method: the latter method provided the eigenvalues, π_{kk} and π_{c2c2}, to be directly incorporated into the Equation-based approach, as illustrated in Figure 2[34]. Below, we develop a *three-step procedure* to derive analytical expressions for *all* elasticities; both for *current demographic changes* and for *expected future demographic changes*, but *not* for the eigenvalues[35]:

[34] The remaining elasticities for all variables in the vector $\widehat{e}_t$ with respect to current shocks could be derived from the matrices $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$. However, we choose to employ the Equation-based method to derive these, because it is less cumbersome to derive analytical expressions for elasticities with this method rather than with the Matrix-based method.

[35] The Equation-based method is relatively similar to the version in Uhlig (1999; section 3.8.3). My version ensures that the model can also be solved with expected future shocks.

1 *The first step* is to take the log-linearised equations and substitute with laws of motion for all variables. If variables enter in period $t+1$ the law of motion is substituted in forwarded form, which requires that one inserts the laws of motion for endogenous state variables. This procedure provides equations where the sum of all exogenous demographic changes, multiplied by a coefficient for each shock will equal zero. Appendix A.3 gives an example with the resource constraint, but this procedure must be applied to all eight log-linearised equations.

2 *The second step* is to collect from all log-linearised equations the coefficients for each *current shock*. As an example this is done in Appendix A.3 with the variable for lagged fertility, $\widehat{b}_{t-1}$. Then solve the equations for the unknown elasticities for current demographic changes. The elasticity of leisure with respect to lagged fertility, π_{lb1}, can then as an example be derived in (39):

$$\pi_{lb1} = [\pi_{c2k} - \pi_{Rk}] \frac{\Lambda_{12}\pi_{wb1} - \Lambda_{21}\pi_{lb1} - \Lambda_8\pi_{\theta b1}}{\Lambda_9\pi_{wk} - \Lambda_7\pi_{c2k} + \Lambda_{12}\pi_{Rk} - \Lambda_{20}\pi_{lk}} + \Lambda_{23}\pi_{\theta b1} - \Lambda_{22}\pi_{wb1} \quad (39)$$

3 *The third step* is to derive elasticities for *expected future* demographic changes. These elasticities depend on current period elasticities, as for instance π_{lb1} in (39). As an example, the coefficients for endogenous variables w.r.t. life expectancy, $\widehat{\phi}^e_t$, are derived in Appendix A.3. The solution is characterized by the necessity of substituting the equilibrium law of motion for endogenous state variables *once more* than for current demographic changes. The elasticity for leisure w.r.t. life expectancy, $\pi_{l\phi e}$, is derived as an example in (40):

$$\pi_{l\phi e} = [\pi_{c2k} - \pi_{Rk}]\,\pi_{kb1} + \Lambda_{23}\pi_{\theta\phi e} - \Lambda_{22}\pi_{w\phi e} + (\pi_{c2\phi e1} - \pi_{R\phi e1}) \quad (40)$$

This completes the presentation of the Matrix-based method in section A.1 and the development of the Equation-based method in section A.2. The combination of these two methods, as illustrated in Figure 1A and described in the *three-step* procedure above, will provide all the elasticities for the recursive equilibrium law of motion for the stochastic OLG economy. The purpose of the following section is to interpret these elasticities, both theoretically and numerically, and to employ them in policy reflections on intergenerational welfare. This involves calibrating the model using what we believe are realistic parameter values, as shown in table 1, and simulating the model using a Matlab routine (available upon request).

A.3 Reduced equations for the solution for elasticities

The resource constraint is used as an example for step 1 in the *Equation-based* method. The log-linearised resource constraint is;

$$\begin{aligned} 0 \;=\; & \Lambda_1\widehat{k}_{t-1} - \Lambda_5\widehat{k}_t + \Lambda_{15}\widehat{l}_t - \Lambda_3\widehat{c}^1_t - \Lambda_4\widehat{c}^2_t \\ & +\Lambda_4\widehat{\chi}_{t-1} - \Lambda_{14}\widehat{\chi}_t - \Lambda_2\widehat{b}_{t-1} - \Lambda_4\widehat{\phi}^e_{t-1} - \Lambda_4\widehat{\phi}^u_t \end{aligned} \quad (41)$$

Example of Step 1 After substituting the laws of motion for all variables the equation is transformed to:

$$
\begin{array}{rll}
0 = & \widehat{k}_{t-1} & \left(\Lambda_1 + \Lambda_{15}\pi_{lk} - \left(\Lambda_5 + \Lambda_3\pi_{c2k} - \Lambda_3\pi_{Rk}\right)\pi_{kk}\right) \\
- & \widehat{\chi}_{t-1} & \left(\Lambda_4\pi_{c2\chi 1} - \Lambda_{15}\pi_{l\chi 1} - \Lambda_4 + \left(\Lambda_5 + \Lambda_3\pi_{c2k} - \Lambda_3\pi_{Rk}\right)\pi_{k\chi 1}\right) \\
- & \widehat{\chi}_{t} & \left(\Lambda_4\pi_{c2\chi} - \Lambda_{15}\pi_{l\chi} + \Lambda_3\pi_{c2\chi 1} - \Lambda_3\pi_{R\chi 1} + \Lambda_{14} + \left(\Lambda_5 + \Lambda_3\pi_{c2k} - \Lambda_3\pi_{Rk}\right)\pi_{k\chi}\right) \\
- & \widehat{b}_{t-1} & \left(\Lambda_4\pi_{c2b1} - \Lambda_{15}\pi_{lb1} + \Lambda_2 + \left(\Lambda_5 + \Lambda_3\pi_{c2k} - \Lambda_3\pi_{Rk}\right)\pi_{kb1}\right) \\
- & \widehat{\phi}_{t}^{u} & \left(\Lambda_4\pi_{c2\phi u} - \Lambda_{15}\pi_{l\phi u} + \Lambda_4 + \left(\Lambda_5 + \Lambda_3\pi_{c2k} - \Lambda_3\pi_{Rk}\right)\pi_{k\phi u}\right) \\
- & \widehat{\phi}_{t}^{e} & \left(\Lambda_4\pi_{c2\phi e} - \Lambda_{15}\pi_{l\phi e} + \Lambda_3\pi_{c2\phi e1} - \Lambda_3\pi_{R\phi e1} + \left(\Lambda_5 + \Lambda_3\pi_{c2k} - \Lambda_3\pi_{Rk}\right)\pi_{k\phi e}\right)
\end{array}
$$

Example of Step 2 As an example of the solution method we solve for the elasticities of endogenous variables with respect to the current change in lagged fertility, $\widehat{b}_{t-1}$. The coefficients from equations like the resource constraint above are collected from all eight log-linearised equations.

$$0 = -\left\{\Lambda_4\pi_{c2b1} - \Lambda_{15}\pi_{lb1} + \Lambda_2 + \left(\Lambda_5 + \Lambda_3\pi_{c2k} - \Lambda_3\pi_{Rk}\right)\pi_{kb1}\right\} \tag{42}$$

$$\begin{aligned} 0 \;=\;& \{\left[\Lambda_9\pi_{wk} - \Lambda_7\pi_{c2k} + \Lambda_{12}\pi_{Rk} - \Lambda_{20}\pi_{lk}\right]\pi_{kb1} + \\ & \Lambda_{12}\pi_{wb1} - \Lambda_{21}\pi_{lb1} - \Lambda_8\pi_{\theta b1}\} \end{aligned} \tag{43}$$

$$0 = \left\{\left[\pi_{c2k} - \pi_{Rk}\right]\pi_{kb1} - \pi_{c1b1}\right\} \tag{44}$$

$$0 = \left\{\pi_{c1b1} + \Lambda_{23}\pi_{\theta b1} - \pi_{lb1} - \Lambda_{22}\pi_{wb1}\right\} \tag{45}$$

$$0 = \left\{\Lambda_{11}\left(\pi_{lb1} - 1\right) - \pi_{yb1}\right\} \tag{46}$$

$$0 = \left\{-\Lambda_{10}\pi_{lb1} - \pi_{Rb1} + \Lambda_{10}\right\} \tag{47}$$

$$0 = \left\{-\pi_{\theta b1} - 1\right\} \tag{48}$$

$$0 = -\left\{\Lambda_{11} + \pi_{wb1} - \Lambda_{11}\pi_{lb1}\right\} \tag{49}$$

The equations (42)-(49) can be solved for the eight unknown elasticities, as for instance the elasticity of leisure with respect to lagged fertility, π_{lb1} (equation 39).

Example of Step 3 As an example of the solution method we solve for the elasticities of endogenous variables with respect to the current change in *life expectancy*, $\widehat{\phi}_{t}^{e}$.

$$0 = -\left\{\Lambda_4\pi_{c2\phi e} - \Lambda_{15}\pi_{l\phi e} + \Lambda_3\pi_{c2\phi e1} - \Lambda_3\pi_{R\phi e1} + \left(\Lambda_5 + \Lambda_3\pi_{c2k} - \Lambda_3\pi_{Rk}\right)\pi_{k\phi e}\right\} \tag{50}$$

$$\begin{aligned} 0 \;=\;& \{\left[\Lambda_9\pi_{wk} - \Lambda_7\pi_{c2k} + \Lambda_{12}\pi_{Rk} - \Lambda_{20}\pi_{lk}\right]\pi_{k\phi e} + \Lambda_{12}\pi_{R\phi e1} - \\ & \Lambda_7\pi_{c2\phi e1} - \Lambda_{20}\pi_{l\phi e1} + \Lambda_9\pi_{w\phi e1} - \Lambda_{12} - \Lambda_{21}\pi_{l\phi e} + \Lambda_{12}\pi_{w\phi e} - \Lambda_8\pi_{\theta\phi e}\} \end{aligned} \tag{51}$$

$$0 = \left\{\left[\pi_{c2k} - \pi_{Rk}\right]\pi_{k\phi e} - \pi_{c1\phi e} + \pi_{c2\phi e1} - \pi_{R\phi e1}\right\} \tag{52}$$

$$0 = \left\{\pi_{c1\phi e} + \Lambda_{23}\pi_{\theta\phi e} - \pi_{l\phi e} - \Lambda_{22}\pi_{w\phi e}\right\} \tag{53}$$

$$0 = \left\{\Lambda_{11}\pi_{l\phi e} - \pi_{y\phi e}\right\} \tag{54}$$

$$0 = \left\{-\Lambda_{10}\pi_{l\phi e} - \pi_{R\phi e}\right\} \tag{55}$$

$$0 = -\left\{\pi_{w\phi e} - \Lambda_{11}\pi_{l\phi e}\right\} \tag{56}$$

$$0 = \left\{-\pi_{\theta\phi e}\right\} \tag{57}$$

The equations (50)-(57) can be solved for the eight unknown elasticities, as for instance the elasticity of leisure with respect to life expectancy, $\pi_{l\phi e}$.

B Calibration of the model

PARAMETER	CALIBRATION	INTERPRETATION
α	1/3	Capital share in output
β	0.30	Replacement rate[36]
a	0.40	Steady state productivity growth rate
δ	1	Rate of depreciation in capital
χ	1	Length of working period
ϕ	1.9	Total length of adult life
λ	$n.a.$	Length of retirement period (*residual*)
b	0.1	Growth rate of the number of children
γ	1	Weight of leisure in utility
$\pi_{\rho_{1(b)}}$	1	Elasticity of the weight of first period consumption in utility with respect to the birth rate
ρ_2	0.296	Consumption discount rate[37]- calibrated so the savings rate is equal to exactly 20%

TABLE 1B. CALIBRATION OF THE MODEL

[36] The payroll tax rate will then be $\theta = \beta\,(\phi - \chi)\,/\,(1 + n^w) = 0.25$.

[37] The calibration of the discount rate equals 0.963 per year or 0.296 over a 30 year period.