Anticipation, Occurrence and Magnitude of Market Crashes

by

Patrick Leoni

Discussion Papers on Business and Economics
No. 12/2008

FURTHER INFORMATION
Department of Business and Economics
Faculty of Social Sciences
University of Southern Denmark
Campusvej 55
DK-5230 Odense M
Denmark

Tel.: +45 6550 3271
Fax: +45 6550 3237
E-mail: lho@sam.sdu.dk
http://www.sdu.dk/osbec

ISBN 978-87-91657-28-3

Anticipation, Occurrence and Magnitude of Market Crashes*

Patrick Leoni

Department of Business and Economics

University of Southern Denmark

Campusvej 55, DK-5230 Odense M, Denmark

E-mail: pal@sam.sdu.dk

*This project was supported by the grant "Complex Markets" during my visit at the GREQAM. I am indebted to P. Gottardi, A. Kirman, O. Chanel and F. Rousseau for insightful conversations. All remaining errors are mine.

Abstract

We simulate a standard Dynamic Stochastic General Equilibrium model to analyze the sensibility of market crashes to the anticipations of endowment drops. Contrary to the commonly accepted view that market crashes are solely driven by large drops in aggregate endowments, we observe in complete markets that: 1- a large and subjective anticipation of an endowment drop amplifies the magnitude of the crash regardless of the level of risk-aversion, and 2- there always exists an upper-bound on the maximal anticipation of the drop so that the crash magnitude remains constant regardless of the drop level. We thus establish that the occurrence and magnitude of a crash in complete markets depend on the anticipation level of the drop, regardless of how the anticipation is formed.

JEL codes: G12, G14.

Keywords: Market crashes, Investors' psychology, Market anticipation.

1 Introduction

Sudden crashes are common features of financial markets. For instance, most of the world' leading market indices lost in 2008 in average 25% of their value in less than a month, whereas the their growth rate was steady since 2003. Another typical example is the 1994 Peso crisis in Mexico, where lending rates rose by four hundred percent over four months. Psychological factors are believed to play an important role in such situations, for instance the 2008 historical crash can be largely imputed to the fear of bankruptcy of some large banks despite the actual economic recession that was about to occur. The actual mechanisms linking such factors and market volatility are not yet fully explored, and this paper par addresses this issue by analyzing the sensibility of market crashes to the subjective anticipation of endowment drops in a standard Dynamic Stochastic General Equilibrium model.

This paper thus investigates the link between market volatility and investors' psychology in a standard general equilibrium with complete markets (see Leoni [7] for a similar analysis and more restrictive results with incomplete markets). Contrary to the commonly accepted view that market crashes are solely driven by large drops in aggregate endowments, we show in a numerical simulation that: 1- a large and subjective anticipation of an endowment drop amplifies the magnitude of the crash regardless of the level of risk-aversion, and 2- there always exists an upper-bound on the maximal anticipation of the drop so that the crash magnitude remains constant regardless of the drop level. We thus establish that the occurrence and magnitude of a crash in complete markets depend on the anticipation level of the drop, regardless of how the anticipation is formed.

Anticipations of changes in fundamentals, driven by psychological factors as described later, are shown here to be a key explanatory factor of market volatility. Our point here is to show that, in a general equilibrium framework, belief-driven variations in demand are a necessary condition for crashes to occur, and that the origins of market crashes stem both from psychological factors and economic factors and not from variations in fundamentals only.

The basic insight of our results is that, when anticipating a future albeit uncertain drop in aggregate endowments, traders take immediate financial positions to hedge against this event. The hedging can only be achieved by purchasing assets paying off positive dividends in this event, thus current purchasing prices are high and in turn returns are low at the time dividends are paid.

In more details, we carry out a set of numerical simulations in the well-known framework of Mehra and Prescott (1985) to make explicit the direct relationship between the actual drop in endowments, the anticipation level of this drop, and the occurrence and magnitude of a market crash. We define an $\varepsilon-$crash (for $\varepsilon > 0$) to be an event where the return of every traded asset paying off positive dividends in this event is below ε. In this setting, we show that for commonly observed levels of endowment drops, the higher the level of anticipation the higher the magnitude of the crash. Highest crash magnitudes are associated with the highest levels of anticipations, and high anticipations significantly intensify the crash magnitudes regardless of the drop level. We also show that, for those same realistic drop levels, there is always an upper-bound on the maximal anticipation of the drop so that the crash magnitude remains constant. This result implicitly shows that a crash of a given magnitude may not occur when its anticipation is low enough, or at least that an otherwise significant market crash may have a significantly lower magnitude when its anticipation is small enough.

Our results rely on the Inada conditions to obtain, as implicitly assumed in the Mehra-Prescott framework, although those conditions alone cannot lead to a crash unless traders largely agree upon a variation in fundamentals. The intuition is that the marginal disutility of a low consumption level on a particular history, typical of Inada conditions, can be compensated in terms of ex-ante utility by a low probability assigned to this history by every agent. Thus in this situation, a low contingent consumption need not be largely hedged against and a crash may not occur.

Our findings are consistent with the commonly observed episodes of market crashes, although our theoretical explanation differs from that in Lee (1998) for instance. Lee justifies crashes by information flows varying with private information, and crashes occur as an informational cascade when enough signals of bad times are released by traders. In contrast, we argue that psychological factors triggering above large anticipations and thus large crashes may also stem from other sources such as herding, market rumors, fear of contagion or panic (or possibly all those issues together, see Shiller, 2000). We point out that all of those factors are all relevant because they may lead to the same phenomenon: a crash anticipation. In this respect, we present a general theory of market crashes where the driving factor of the occurrence and the magnitude of a crash is its anticipation, regardless of how the anticipation is formed. The coordination among many traders needed to form a large enough anticipation may thus stem from many other sources such as rational expectations and erratic beliefs (a point thus consistent with Allen *et al.* (2005) for instance), although it contains as a particular case the situation raised in Lee (1998) above and Ho and Stein (2003).

2 The model

In this section, a formal description of the model is given. Time is discrete and continues forever. In every period $t \in N_+$, a state is drawn by nature from a set $S = \{1, ..., L\}$, where L is strictly greater than 1. Before defining how nature draws the states, we first need to introduce some notations.

Denote by S^t ($t \in N \cup \{\infty\}$) the $t-$Cartesian product of S. For every history $s_t \in S^t$ ($t \in N$), a cylinder with base on s_t is defined to be the set $C(s_t) = \{s \in S^\infty | \ s = (s_t, ...)\}$ of all infinite histories whose t initial elements coincide with s_t. Define the set Γ_t ($t \in N$) to be the $\sigma-$algebra which consists of all finite unions of cylinders with base on S^t.[1] The sequence $(\Gamma_t)_{t \in N}$ generates a filtration, and define Γ to be the $\sigma-$algebra generated by $\cup_{t \in N} \Gamma_t$. Given an arbitrary probability measure Q on (S^∞, Γ), we define $dQ_0 \equiv 1$ and dQ_t to be the Γ_t-measurable function defined for every $s_t \in S^t$

[1]The set Γ_0 is defined to be the trivial $\sigma-$algebra, and $\Gamma_{-1} = \Gamma_0$.

$(t \in N_+)$ as

$$dQ_t(s) = Q(C(s_t)) \text{ where } s = (s_t, ...).$$

Given data up to and at period $t-1$ $(t \in N)$, the probability according to Q of a state of nature at period t, denoted by Q_t, is

$$Q_t(s) = \frac{dQ_t(s)}{dQ_{t-1}(s)} \text{ for every } s \in S^\infty,$$

with the convention that if $dQ_{t-1}(s)$=0 then $Q_t(s)$ is defined arbitrarily.

In every period and for every finite history, nature draws a state of nature according to an arbitrary probability distribution P on (S^∞, Γ). To simplify the analysis, we assume that $P_{s_t} > 0$ for every history s_t.

To conclude this section, we define the operators E^Q to be the expectation operator associated with Q. Finally, we say that a finite history $s_{t+p} \in S^{t+p}$ *follows* a finite history $s_t \in S^t$ $(t, p \in N)$, denoted by $s_{t+p} \hookrightarrow s_t$, if there exists $s \in S^p$ such that $s_{t+p} = (s_t, s)$.

2.1 The agents

In this section, economic agents are described. There is a finite number $I \geq 1$ of infinitely-lived agents behaving competitively.

There is a single consumption good available in every period t $(t \in N_+)$. Denote by $c^i_{s_t}$ the consumption of an agent i $(i = 1, ..., I)$ in history $s_t \in S^t$ $(t \in N_+)$. In every period t $(t \in N_+)$ and in every history $s_t \in S^t$, every agent i $(i = 1, ..., I)$ is endowed with $w^i_{s_t} > 0$ units of consumption goods.

In every period $t \in N$, and after the realization of the history $s_t \in S^t$, the agents trade $L \geq 1$ infinitely-lived assets, or Lucas' trees as in Lucas (1978). Every tree j $(j = 1, ..., L)$ yields a dividend $d^j_{s_t} > 0$ of units of consumption good in history s_t. Let d_{s_t} denote the vector $(d^1_{s_t}, ..., d^L_{s_t})$ for every s_t. The supply of every tree is assumed to be 1 in every history.

The aggregate endowment w_{s_t}, in every history s_t ($s_t \in S^t$ and $t \in N_+$), is given by

$$w_{s_t} = \sum_i w^i_{s_t} + \sum_j d^j_{s_t}.$$

The price in history s_t of one share of the tree j $(1 \leq j \leq L)$ is denoted by $q^j_{s_t}$, for every $s_t \in S^t$ and $t \in N_+$. Let q_{s_t} denote the vector $(q^1_{s_t}, ..., q^L_{s_t})$ for every history s_t.

A *portfolio* θ^i for every agent i is a vector $(\theta_{s_t})_{s_t \in S^t, t \in N_+}$ of shares held of the J trees in every history s_t, where $\theta_{s_t} = (\theta^j_{s_t})_j$ is the vector of holdings in history s_t and $\theta^j_{s_t}$ is the holding of j in this same history s_t. Every agent i has an initial portfolio θ^i_0 at date 0.

Every agent i does not have any information about P, the true probability measure from nature draws the states; however agent i has a subjective belief about nature represented by a probability measure P^i on (S^∞, Γ). We assume that $dP^i_t(s) > 0$ for every infinite history s and every period t, to avoid problems of existence as pointed in Araujo and Sandroni (1999). This assumption is not restrictive in practice, since even highly unlikely events can always be regarded as assigned arbitrarily low but positive by every agent.

Every agent derives some utility in any history from consuming the only consumption good present in the economy. We assume that agent i ranks all the possible future consumption sequences $c = (c_{s_t})_{s_t \in S^t, t \in N_+}$ according to the utility function

$$U^i(c) = E^{P^i}\left(\sum_{t \in N_+} (\beta_i)^t u_i(c_t)\right), \tag{1}$$

where $\beta_i \in (0,1)$ is the intertemporal discount factor, u_i is a strictly increasing, strictly concave and continuously differentiable function. We assume that u^i satisfies the Inada condition, namely $(u_i)'(c) \mapsto \infty$ as $c \mapsto 0$.

The budget constraint faced in every history s_t by agent i is

$$\begin{aligned} c_{s_t} + \sum_j q^j_{s_t}\theta^j_{s_t} &\leq w^i_{s_t} + \sum_j d^j_{s_t}\theta^j_{s_{t-1}} + \sum_j q^j_{s_t}\theta^j_{s_{t-1}} && (2) \\ c_{s_t} &\geq 0, && (3) \end{aligned}$$

where $s_t \hookrightarrow s_{t-1}$. The left-hand side of (2) represents the purchase of consumption good at price normalized to 1 plus the purchase of new shares of trees at current prices, and the right-hand side is the endowment plus the dividends payments from previous holdings plus the proceeds from selling the current holdings of trees at current prices.

Given the constraints faced by the traders, we also need to rule out the possibility of rolling over any debt through excessive future borrowing, also known as Ponzi's scheme. Consider any vector of prices that is *arbitrage-free*. As argued in Hernandez and Santos (1996), when a vector of prices q is arbitrage-free there exists a sequence of positive numbers $\{\pi_{s_t}\}_{s_t \in S^t, t \in N_+}$

with $\pi_{s_0} = 1$ such that

$$\pi_{s_t} q^j_{s_t} = \sum_{s \hookrightarrow s_t} \pi_s d^j_s,$$

for every j ($j = 1, ..., J$) and s_t ($s_t \in S^t$ and $t \in N_+$). We now assume that every agent cannot borrow more than the present value of her current endowment at such prices. Formally, we assume that for any vector of prices q that is arbitrage-free, every portfolio strategy satisfies the *wealth constraints*

$$q_{s_t}\theta_{s_t} \geq -\frac{1}{\pi_{s_t}} \sum_{s_\tau \in C(s_t)} \pi_{s_\tau} w^i_{s_\tau} \text{ for every } s_t. \tag{4}$$

This constraint naturally rules out Ponzi's scheme, and it is chosen arbitrarily among many others. Hernandez and Santos (1996) gives six other constraints ruling out Ponzi's schemes and shows that they are all equivalent when markets are complete. Markets will be assumed to be complete throughout this paper, thus any of the constraints given in this last reference can be used in our setting (see Leoni [7] for a similar analysis where markets are incomplete).

For every i, we define the budget set $B^i(q)$ faced by agent i at prices q as follows. If q is arbitrage-free, the budget set $B^i(q)$ is the set of sequences (c, θ) that satisfy conditions (2)-(4) above. If now the vector of prices has an arbitrage opportunity, define $B^i(q)$ as the set of sequences (c, θ) that satisfy conditions (2)-(3) only.

Definition 1 *An equilibrium is a sequence* $(\bar{c}^i, \bar{\theta}^i)_i$ *and a system of prices* $\bar{q}$ *such that*

1. *taking prices $\bar{q}$ as given, for every i the vector $(\bar{c}^i, \bar{\theta}^i)$ is solution to the program consisting of maximizing (1) subject to $(c, \theta) \in B^i(\bar{q})$, and*

2. *for every history s_t we have that $\sum_i \bar{c}^i_{s_t} = w_{s_t}$ and $\sum_i \bar{\theta}^i_{s_t} = 1$.*

The above definition requires that, taking prices as given, every agent sequentially chooses consumption plans and portfolio holdings so as to maximize her expected utility, and markets for consumption good and trees all clear in every history. It is also straightforward to see that the equilibrium prices are arbitrage-free. Indeed, if otherwise then every agent will have an infinite demand for at least one tree in at least one history, and Condition 2 in the above definition will always be violated. By a similar reasoning, it is easy to check that equilibrium prices must be strictly positive.

2.2 Market crashes

We next describe the notion of *market crashes* occurring in financial markets. This notion focuses on arbitrarily low returns on traded trees. For every system of asset prices q, define first the *return* of tree j ($j = 1, ..., J$) in history s_{t+1}, when purchased in history s_t, as

$$R^j_{s_{t+1}} = \frac{q^j_{s_{t+1}} + d^j_{s_{t+1}}}{q^j_{s_t}} \tag{5}$$

if $q^j_{s_t} > 0$, and arbitrarily otherwise. With this notion, we can describe our notion of market crash.

Definition 2 *For every $\varepsilon > 0$, an ε-crash occurs in history s_t if $R^j_{s_t} < \varepsilon$ for every asset j such that $d^j_{s_t} > 0$.*

A market crash in a given history is thus defined as an arbitrarily low return on every asset paying off strictly positive dividend in this history. In the remainder of the paper, we are primarily interested in finding conditions leading to arbitrarily low market crashes. In particular, we analyze how individual anticipations of variations in market fundamentals can generate crashes as described above.

3 Amplification of crashes

We now carry out some numerical simulations to find anticipation levels sustaining arbitrary levels of crashes. We narrow down our model to that in Mehra and Prescott (1985), with the difference that we do not assume any condition on the endowment process and we allow for arbitrary beliefs. Our first simulation gives a region for the parameters δ and γ sustaining a given crash magnitude for various levels of risk-aversion. The second simulation shows that, for a given level of endowment drop this time, the higher the anticipation the higher the crash magnitude. The third simulation is a 3D-representation of crash magnitudes as a function of both drops and anticipations, illustrating the intuitions given in the Introduction.

We now assume, following Mehra and Prescott (1985), that in every period two states only can occur. We also assume that there is one agent only within the economy (a representative agent) forming subjective belief about economic uncertainty. Even if strong in appearance, this last assumption has already been largely justified in terms of macroeconomic analysis. The representative agent has a utility function of the form

$$U(c) = E^P \left(\sum_{t \in N_+} \beta^t u(c_t) \right),$$

where P is an arbitrary belief process, where $\beta \in (0, 1)$ is a constant, and where the function u is defined as

$$u(x) = \frac{x^{1-\alpha} - 1}{1 - \alpha},$$

for some $\alpha > 0$ (this parameter is the coefficient of risk-aversion of the agent).

Fix now any history s_{t-1}, let $\bar{s}_t \hookrightarrow s_{t-1}$ be the history following s_{t-1} where the crash is expected and let $s_t \hookrightarrow s_{t-1}$ be the other history following s_{t-1}. In Appendix A, we show that

$$R^j_{\bar{s}_t} \leq \frac{1}{\beta} \cdot \frac{1}{P_{\bar{s}_t}} \cdot \left(\frac{w_{\bar{s}_t}}{w_{s_{t-1}}} \right)^\alpha \tag{6}$$

for every security j as before, and regardless of the asset structure provided that the agent is not constraint in borrowing in equilibrium. In particular, Inequality (6) shows that the upper-bound on equilibrium returns depends only the parameters γ, δ, α and β. The following numerical simulations are generated directly from this last inequality.

From now on, we fix $\beta = 0.9$ since this parameter does not a critical role in our analysis. Our first simulation provides a parameters region sustaining a .85-crash, which corresponds to a drop of 15% in price of all assets traded (assuming no dividend is paid).

Figure 1 simultaneously displays such regions for various level of risk-aversion. For every curve, any point of parameters above the curve sustains

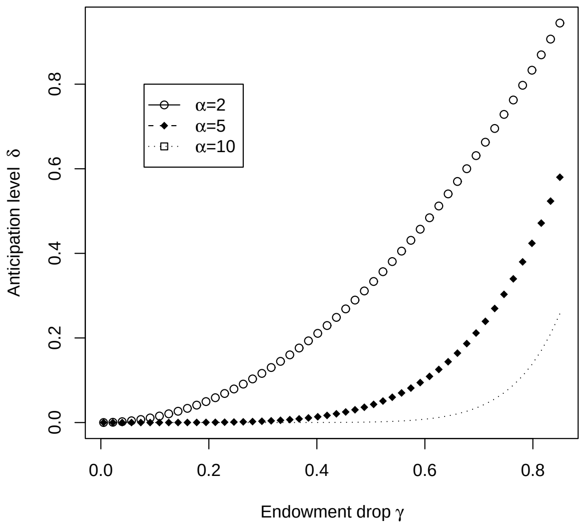

Figure 1: Parameters region sustaining a .85-crash (15% price drop)

the .85-crash. For instance, for an agent with a level of risk-aversion of 5, any 20% drop in endowment next period that is anticipated with probability of at least .5 in the current period will trigger a .85-crash next if the drop actually occurs. Figure 1 also shows that, for those last values, any anticipation level below .5 may not trigger the crash, as is explained in the Introduction. This last point implies that the crash occurs independently of the true probability of a drop next period, showing that the anticipation (together with the drop of course) has driven the crash.

The next figure gives us a way to visualize the effect of drop anticipations on the magnitude of a crash, given a particular drop of endowment next period. We fix a 20% drop in the following simulation.

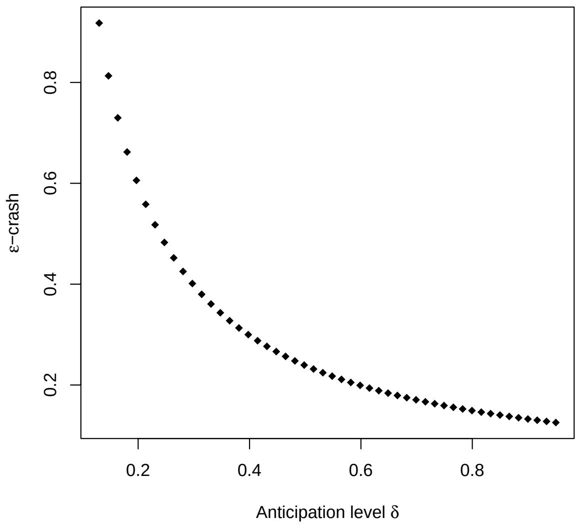

Figure 2: Crash magnitude as a function of the anticipation δ ($\alpha = 10$)

Figure 2 provides the direct link between the magnitude of the crash and the anticipation of the drop. Its main implication is that, for a fixed drop of endowment, the higher the anticipation the higher the crash magnitude. The intuition of this point is also given in the Introduction.

Figure 3 below maps crash magnitudes as a function of both the anticipation levels and endowments drops. Regions of relatively low endowment drops and low anticipations leads to moderate crashes. Regions of high anticipations of drops trigger the highest levels of crash, and such high anticipations significantly intensify the crash magnitudes regardless of the drop level. Provided that anticipations are high enough, severe drops in endowment lead to severe crashes (as is commonly believed), but our point is to show that

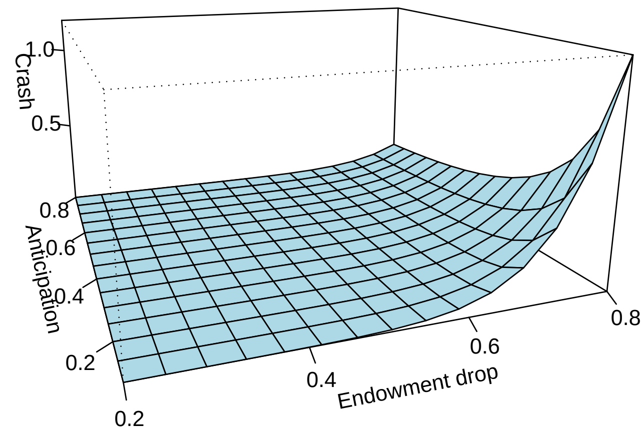

Figure 3: ϵ-crash as a function of the anticipation and drop ($\alpha = 10$)

anticipations do intensify this phenomena. That is, psychological factors as described here turn crashes from bad to significantly worse.

4 Occurrence of crashes

We now determine under which conditions a low enough anticipation of an endowment drop can decrease the magnitude of the crash. To simplify matters, we narrow down without loss of generality our asset structure to Arrow securities, defined as securities that pay off one unit of consumption good next period if a particular state occurs and 0 otherwise. We will consider a set of those securities so that markets are complete; this approach is legitimate and without loss of generality since as explained later the price of any original security can be expressed as a combination of the price of those Arrow securities when markets are complete.

In more details, we consider the following asset structure. In every period $t \in N$, and after the realization of the history $s_t \in S^t$, the agent trades $L = 2$ one-period securities. Every security $a^j_{s_t}$ ($1 \leq j \leq L$ and $s_t \in S^t$) purchased in history s_t payoffs $a^j_{s_t}(l)$ unit of consumption good in period $t+1$ if state l is drawn, where $a^j_{s_t}(l) = 1$ if $j = l$ and 0 otherwise. Such securities are commonly known as Arrow securities. The choice of this asset structure is solely meant to ensure that, for every vector of strictly positive prices, markets are complete. Let a_{s_t} denote the vector $(a^1_{s_t}, ..., a^L_{s_t})$ for every s_t. The supply of each security is assumed to be 0 in every history.

Since we focus on this asset structure only, the price in history s_t of security $a^j_{s_t}$ ($1 \leq j \leq L$) is still denoted by $q^j_{s_t}$, for every $s_t \in S^t$ and $t \in N_+$. Let q_{s_t} denote the vector $(q^1_{s_t}, ..., q^L_{s_t})$ for every history s_t.

With this asset structure in complete markets, Huang and Werner [3] shows that the equilibrium price of any security j as described in the previous section can be rewritten in every history s_t as

$$q^j_{s_t} = \sum_{s \in C(s_t)} d^j_s \cdot a^j_s. \tag{7}$$

It is therefore sufficient to analyze market crashes with Arrow securities, and to recast the conditions on occurrence for our original securities following Eq.

(7).

The simulations in this section are based on the following equality, which is proved in Appendix B. Denote by $\varepsilon > 0$ the magnitude of the crash and by $\gamma > 0$ the percentage of the endowment drop, we then have for those Arrow securities that

$$P_{s_t} = \frac{1}{\varepsilon} \cdot \frac{\gamma^2}{\beta}. \tag{8}$$

Equality (8) explicitly gives for Arrow securities an upper-bound of the anticipation level that can sustain in equilibrium a crash of magnitude $\varepsilon > 0$ for an endowment drop γ. Any anticipation below the right-hand side of (8), given γ, must lead to a market crash of magnitude at most ε.

We first seek to identify the maximal anticipation level that can generate at most a crash of a given magnitude, for every possible level of endowment drop. The point is to find a relationship between endowment drops and anticipation so that the magnitude of the crash remains unaffected by psychological factors. Fig. 4 shows the results of our simulation when considering a 15% price drop or .85-crash, which corresponds to severe albeit already observed historical events such as the Black Monday in 1987.

We observe that the maximal anticipation level is an increasing function of the endowment drop. In other words, we have the natural property that the higher the endowment drop, the higher the maximal anticipation to secure a crash of at most 15%. The convexity of the functions plotted depends on the risk aversion level. This fact is explained by the fact that the more risk-averse the agent the lower the maximal anticipation level. In this case, risk

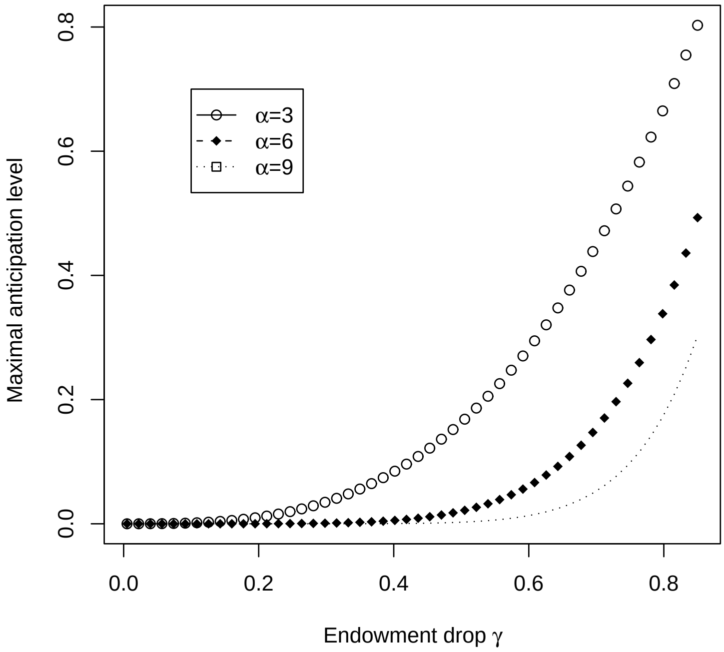

Figure 4: Maximal anticipation level as function of endowment drops that cannot generate a crash of more than 15%, for various levels of risk aversion.

aversion alone forces the agent to over-invest in the Arrow corresponding to the future event when the crash may occur, leading in turn to a lower return, without requiring a large anticipation of the drop.

We now turn to analyzing the maximal anticipation level that can sustain any possible magnitude of crash, for a fixed level of endowment drop. Our point is to find the frontier of highest anticipation for a crash magnitude, given a shock in fundamental. Fig. 4 shows the results of our simulation when considering an endowment drop of 5%, which corresponds also to already observed historical recessions.

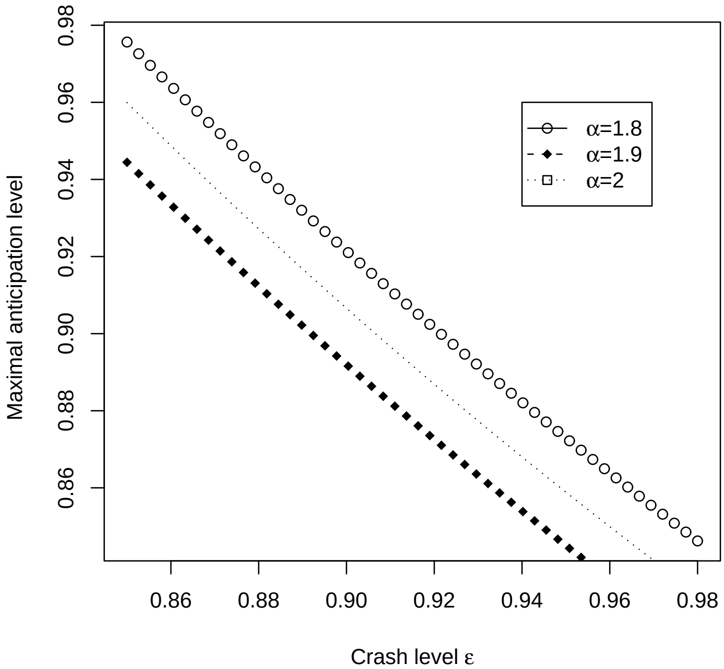

Figure 5: Maximal anticipation levels to generate corresponding crash magnitudes, for a given 5% endowment drop and for various levels of risk aversion.

The simulation shows that the maximal anticipation for this given 5% endowment drop is a decreasing function of the crash level. In other words, we observe that the higher the crash level the lower the maximal anticipation necessary to sustain this crash level. This result stems essentially from the fact that a large range of beliefs can prevent low market crashes, but to prevent large crashes from occurring it takes increasingly low anticipation beliefs. Risk-aversion plays a similar role as in the previous simulation in explaining the decrease in maximal anticipation as the risk aversion increases. An increase in risk aversion results in increasing over-investments in the corresponding Arrow security paying off when the drop occurs, and therefore the attenuating effects of the maximal anticipation are decreasing accordingly.

5 Conclusion

We have simulated a standard Dynamic Stochastic General Equilibrium model to analyze the sensibility of market crashes to the anticipations of endowment drops. Our main finding is that, in complete markets, a large anticipation of an endowment drop amplifies the magnitude of the crash. This phenomenon is present regardless of the level of risk aversion of the agents, and it appears as a sole consequence of beliefs effects. The basic insight is that, when expecting future low endowments, agents will increase their demand for securities to hedge against this event. This, in turn, will raise the purchasing price of those securities and therefore will lower their returns.

This last intuition also explains our final findings. We have observed that, regardless of the drop in endowments, there is always an upper-bound on the maximal anticipation of the drop so that the crash magnitude remains constant. In other words, the magnitude of the crash is always lowered when the drop is not largely expected. In this case of unanticipated drop, investors purchase few hedging products against this drop, and thus the purchasing prices remain low enough to control the magnitude of the crash.

The psychological factors at the very heart of those critically important anticipations can be rather arbitrary in our study. We point out that any factors such as for instance herding, market rumors, fear of contagion or panic (and possibly all those issues together) may trigger the anticipation of the drop and thus may magnify the magnitude of the crash. We do not sort which one seems most likely, but rather we point out that they are all relevant because they may lead to the same anticipation. In this respect, we present a general theory of market crashes where the driven factor of the occurrence and the magnitude of a crash is its anticipation, regardless of how the anticipation is formed.

A Proof of Inequality (6)

We next prove this inequality central to our first set of numerical simulations.

Consider the original program of any agent i, consisting of maximizing (1) subject to $(c, \theta) \in B^i(\bar{q})$ and taking as given any arbitrage-free and strictly positive asset prices. Since we assume that Constraint (4) does not bind, and since we know by the Inada conditions that Constraint (3) does not bind as well, the Lagrangian to this program rewrites as

$$\mathcal{L} = \sum_{s_t} dP^i_{s_t} \beta^t_i u_i(c_{s_t}) + \sum_{s_t} \mu_{s_t} \left[w^i_{s_t} + \sum_j d^j_{s_{t-1}} \theta^j_{s_t} - c_{s_t} + \sum_j q^j_{s_t} (\theta^j_{s_t} - \theta^j_{s_{t-1}}) \right],$$

where for every history s_t the real number $\mu_{s_t} > 0$ is the Lagrange multiplier associated with the Constraint (2). Taking the first-order conditions with respect to every variable yields the following relationships for every history s_{t-1} and asset j

$$\begin{aligned} dP^i_{s_{t-1}} \cdot \beta_i^{t-1} \cdot u'_i(c_{s_{t-1}}) &= \mu_{s_{t-1}} \text{ and} &(9)\\ \sum_{s_t \hookrightarrow s_{t-1}} \mu_{s_t} \cdot [d^j_{s_t} + q^j_{s_t}] &= \mu_{s_{t-1}} \cdot q^j_{s_{t-1}}, &(10)\end{aligned}$$

Rearranging terms gives

$$\sum_{s_t \hookrightarrow s_{t-1}} dP^i_{s_t} \cdot \beta_i^t \cdot u'_i(c_{s_t}) \cdot [d^j_{s_t} + q^j_{s_t}] = dP^i_{s_{t-1}} \cdot \beta_i^{t-1} \cdot u'_i(c_{s_{t-1}}) \cdot q^j_{s_{t-1}}, \tag{11}$$

and by (5) and some simplifications we obtain the desired relationship

$$\sum_{s_t \hookrightarrow s_{t-1}} P^i_{s_t} \cdot \beta_i \cdot u'_i(c_{s_t}) \cdot R^j_{s_t} = u'_i(c_{s_{t-1}}). \tag{12}$$

We now derive an upper-bound on equilibrium returns under the assumptions of Section 4. This uniform upper-bound readily allows for the numerical simulations given in this last section.

Fix any history s_{t-1}, and let $\bar{s}_t \hookrightarrow s_{t-1}$ and $s_t \hookrightarrow s_{t-1}$ be defined as in Section 4. Consider any security j such that Equation (12) holds for those histories. Given the shape of our utility function, and since the consumption of the representative agent must be the aggregate endowment in every history, Equation (12) rewrites as

$$P_{\bar{s}_t} \left(\frac{1}{w_{\bar{s}_t}} \right)^{\alpha} R^j_{\bar{s}_t} + (1 - P_{\bar{s}_t}) \left(\frac{1}{w_{s_t}} \right)^{\alpha} R^j_{s_t} = \frac{1}{\beta} \left(\frac{1}{w_{s_{t-1}}} \right)^{\alpha}, \tag{13}$$

for every security j as described above. Rearranging terms gives

$$R^j_{\bar{s}_t} = \frac{1}{P_{\bar{s}_t}} (w_{\bar{s}_t})^{\alpha} \left[\frac{1}{\beta} \left(\frac{1}{w_{s_{t-1}}} \right)^{\alpha} - (1 - P_{\bar{s}_t}) \left(\frac{1}{w_{s_t}} \right)^{\alpha} R^j_{s_t} \right]. \tag{14}$$

Moreover, since in equilibrium it must be true that $R^j_{s_t} > 0$, we obtain the following inequality

$$R^j_{\bar{s}_t} \leq \frac{1}{\beta} \cdot \frac{1}{P_{\bar{s}_t}} \cdot \left(\frac{w_{\bar{s}_t}}{w_{s_{t-1}}} \right)^{\alpha} \tag{15}$$

for every security j as above. The right-hand side of Inequality (15) depends on the parameters described in Proposition 3, together with the intertemporal discount factor β and the coefficient of risk-aversion α. This last inequality directly yields the desired inequality.

B Proof of Eq. (8)

We now turn to a setting where the only traded securities are Arrow securities, without loss of generality by Eq. (7). Given the payoff pattern of those securities, Eq. (13) rewrites as

$$P_{\bar{s}_t} \left(\frac{1}{w_{\bar{s}_t}} \right)^{\alpha} R^j_{\bar{s}_t} = \frac{1}{\beta} \left(\frac{1}{w_{s_{t-1}}} \right)^{\alpha}, \tag{16}$$

where $\bar{s}_t = (s_{t-1}, j)$, thus the equality above is true only for the only Arrow security with payment in state (s_{t-1}, j). Rearranging the previous equation gives

$$P_{\bar{s}_t} = \frac{1}{\beta} \cdot \frac{1}{R^j_{\bar{s}_t}} \cdot \left(\frac{w_{\bar{s}_t}}{w_{s_{t-1}}} \right)^{\alpha}. \tag{17}$$

Therefore in equilibrium, for a given crash level $\varepsilon > 0$ and a given endowment drop $\gamma > 0$, the only anticipation level that can sustain those values is given by

$$P_{\bar{s}_t} = \frac{1}{\beta} \cdot \frac{1}{\varepsilon} \cdot \gamma^{\alpha}, \tag{18}$$

which corresponds exactly to the desired Eq. (8)

References

[1] Allen, F., Morris, S., and H.Y. Shin, 2005, "Beauty contests and bubbles," forthcoming in *Review of Financial Studies*.

[2] Araujo, A., and A. Sandroni, A. (1999) "On the Convergence to Homogeneous Expectations when Markets are Complete," *Econometrica*, 67, 663-672.

[3] Huang, K., Werner, J. (2004) Implementing Arrow-Debreu equilibria by trading infinitely-lived securities. *Economic Theory* 24, 603-623.

[4] Hong, H., and J. Stein (2003) "Differences in Opinion, Short-Sales Constraints and Market Crashes," *Review of Financial Studies*, 16, 487-525.

[5] Hernandez, A., and M. Santos (1996) "Competitive Equilibria for Infinite-horizon Economies with Incomplete Markets, *Journal of Economic Theory*, 71, 102-130.

[6] Lee, I. H. (1998) "Market crashes and informational avalanches," *Review of Economic Studies*, 4, 741-759.

[7] Leoni, P. (2008) "Psychological aspects of market crashes," forthcoming in ICFAI Journal of Behavioral Finance.

[8] Lucas, R. (1978) "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429-1445.

[9] Mehra, R., and E. Prescott (1985) "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15, 145-161.

[10] Shiller, R. (2000) *Irrational Exuberance.* Princeton University Press, Princeton.