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Discussion Papers on Business and Economics  
No. 1/2009

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ISBN 978-87-91657-29-0

# LABOUR SUPPLY AND RETIREMENT POLICY IN AN OVERLAPPING GENERATIONS MODEL WITH STOCHASTIC FERTILITY\*

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December 19, 2008

## Abstract

Using a stochastic general equilibrium model with overlapping generations, this paper studies a policy rule for the retirement age aiming at offsetting the effects on the supply of labour following fertility changes. We find that the retirement age should increase more than proportionally to the direct fall in the labour supply caused by a fall in fertility. The robustness of this result is checked against alternative model specifications and parameter values. The efficacy of the policy rule depends crucially on the link between the preference for leisure and the response of the intensive margin of labour supply to changes in the statutory retirement age.

*JEL Classification:* D91; E20; H55; J10; J26

*Keywords:* Labour supply; fertility; retirement age; overlapping generations; method of underdetermined coefficients.

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\*For valuable comments to this paper, we would like to thank David Bloom, Guenther Fink, Martin Flodén, Bo Sandeman Rasmussen, Lars Lønstrup and Casper Hansen.

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# 1 Introduction

As a result of low fertility rates since the 1960s, the labour supply in most developed countries is likely to shrink over the coming decades. Since a smaller labour force increases the capital-labour ratio, an upward pressure on wages is likely to arise, so the work-leisure choice becomes important.<sup>1</sup> In fact, workers might react quite strongly to a higher opportunity cost of leisure by demanding less of it, thus countering the effects of the fertility fall. However, with a strong income effect, the net effect of higher wages might be an increased demand for leisure, so the effective supply of labour would be reduced even further.

In this paper we present a model that captures these economic relationships. The analytical framework is a dynamic stochastic general equilibrium (DSGE) model with overlapping generations. In a three-period setup, we formulate the explicit relationship between the extensive and the intensive margin of labour supply, where the statutory retirement age functions as a proxy for the extensive margin. In line with Chakraborty (2004), we make the length of the retirement period residually determined by, first, the length of the working period and, second, the total length of life. The novelty of this approach is the feasibility of deriving the implications on the intensive margin of labour supply to a change in the statutory retirement age.

The analytical framework is augmented by a policy rule for the statutory retirement age that effectively counteracts the decline in the size of the labour force caused by a lower fertility rate. Indeed, a straightforward policy response would be to increase the statutory retirement age in order to retain workers in the labour force for a longer period of time. The paper studies how such a link between changes in fertility and the statutory retirement age can be established. Our approach facilitates an analytical presentation, where the role of each model parameter can be identified. For example, analytical expressions can be derived that links the effect on leisure to changes in, respectively, the statutory retirement age, the fertility rate and the preference for leisure.

By assuming that the statutory retirement age is under government control, it is possible to analyse the *offsetting* response of the statutory retirement age under alternative demographic and economic contingencies. Our main result is that an increase in the retirement age has the potential to offset the (fertility induced) decline in the labour force, provided that the retirement age increases *more* than proportionally to the fall in fertility. The reason is that workers substitute for leisure both when fertility falls and the retirement age increases. Furthermore, an implication of the increase in the retirement age is that lifetime leisure will fall, and this further increases the demand for leisure during the working period. As a result, labour supply will fall not only due to low fertility but also as a side effect of the increase in the retirement age. Consequently, policy makers should account for this endogenous response of labour supply when formulating the optimal policy to alleviate the impact of low fertility. In this context, we identify the crucial link between the preference for leisure and the offsetting response of the statutory retirement age: the higher the preference for leisure the more the statutory retirement

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<sup>1</sup>In fact, Weil (2006) finds that the distortion created by taxes needed to fund PAYG pension systems is a key channel through which a higher dependency ratio affects aggregate output and welfare.

age has to increase in order to offset the fall in the size of the labour force. The intuition behind this result is due to our unique formulation of the relationship between the extensive and intensive margins of labour supply.

In the next section we develop the model and the analytical solution method. Section 3 presents the market equilibrium where we analyse the impacts on key variables of changes in fertility and the statutory retirement age, respectively. Section 4 considers the policy option of changing the statutory retirement age in order to offset the decline in the labour force and provides robustness analyses. Finally, section 5 concludes and outlines a number of potential extensions of the research on this topic.

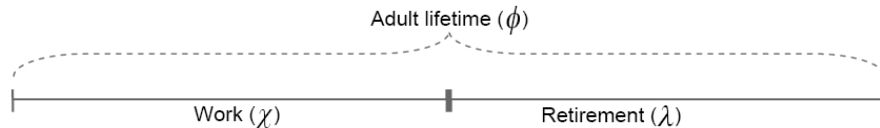
## 2 The Model

In this section we outline a stochastic OLG model with endogenous labour supply. Our model is inspired by Chakraborty (2004), who incorporates the length of life into a standard OLG model with exogenous labour supply *ad modum* Diamond (1965)<sup>2</sup>. The model consists of different building blocks: demographics, households, production, and social security. We present these in turn, before outlining the solution method.

### 2.1 Demographics

Individuals are assumed to be identical across cohorts, and to live for three periods: as children, adults and elderly, respectively. We denote the children born in period  $t$  as  $N_t^c$ , where  $N_t^c = b_t N_t^w$  and  $b_t > 0$  is the birth rate. Adults are denoted by  $N_t^w$  and they are assumed to work for the full length of period  $t$ . During period  $t + 1$  they are retired. The growth rate of the labour force is  $1 + n_t^w = N_t^w / N_{t-1}^w$ , where  $n_t^w \equiv \chi_t b_{t-1}$  is the (net) growth rate of the labour force. The factor  $\chi_t$  denotes the length of the working period, as illustrated in figure 1.

Figure 1. Adult lifetime: work and retirement



If there is an increase in  $\chi_t$ , workers would have to remain in the labour force for a longer time-period. Hence, the effective growth rate of the labour force increases. Also, a fall in the fertility rate in the former period implies a shrinking labour force in the present period. Workers are assumed to elastically supply labour,  $L_t$ , up to one unit,  $u \in (0, 1)$ , where  $L_t = u_t N_t^w$ , and  $u_t$  is the intensity of labour supply in the working period. First period leisure therefore equals  $l_t = 1 - u_t$ .

<sup>2</sup>Bohn (2001) develops a stochastic version of the model, and incorporates the length of the retirement period in a similar way as Chakraborty (2004) models the length of life.

The aggregate measure of adult lifetime,  $\phi$ , is the sum of the lengths of the working period and the retirement period, respectively. Thus, retirement,  $\lambda \in (0, 1)$ , is residually determined as:

$$\lambda_t = \phi_t - \chi_{t-1} \quad (1)$$

More specifically, the total length of life,  $\phi$ , comprises an expected component and an unexpected component, i.e.  $\phi_t = \phi_{t-1}^e \phi_t^u$ , where  $\{\phi\} \in (0, 2)$ . We assume that these components  $\phi_t$  are stochastic and identically and independently distributed. The same is the case for  $b_t$  and  $\chi_t$ . Changes in  $\phi$  will exclusively affect  $\lambda$  if  $\chi$  remains constant. Based on this formulation, we assume that an increase in  $\chi$  is equivalent to an increase in the retirement age. Note that changes in effective labour supply can therefore be decomposed into three effects: first, the effect from the *exogenous extensive* margin,  $\chi$ ; second, the effect from the *endogenous intensive* margin,  $u_t$ ; and, third, the effect from the exogenous *growth* in the number of workers,  $b_{t-1}$ . It is common in the literature to endogenise the intensity of labour supply, but to combine this with changes in labour supply at the extensive margin has, to our knowledge, not previously been attempted<sup>3</sup>.

## 2.2 Households

We adopt a log-utility function, displaying homothetic preferences over consumption and leisure, bearing in mind the well-known limitations of the log-specification<sup>4</sup>.

$$u_t = \chi_t \rho_{1(b_t)} \ln c_{1t} + \chi_t \beta \ln \left( \frac{l_t}{\chi_t} \right) + \rho_2 E_t [\lambda_{t+1} \ln c_{2t+1}] \quad (2)$$

We denote  $c_{1t}$  and  $c_{2t+1}$  as first and second period consumption, respectively. The discount rate on  $c_{2t+1}$  is  $\rho_2 > -1$ , and  $\beta > 0$  is the relative weight on leisure in utility. Decisions about consumption for children are assumed to be made by parents<sup>5</sup>.

Second period consumption is scaled by the length of the retirement period<sup>6</sup>. The higher is  $\lambda$ , the longer period of time retirees can enjoy consumption. While the same argument also applies to the length of the first period,  $\chi$ , for consumption and leisure, we stress that if  $\chi$  increases then some of the "sub-periods" in retirement,

<sup>3</sup>As a result, if fertility falls by, for instance, 1% and the response by the government is to increase the statutory retirement age by 1% both these events may lead to an increased demand for leisure by households and thus a fall in labour supply. The 1% increase in the retirement age may therefore not be enough to counteract the general equilibrium effects that induce workers to reduce their labour supply. This result is clearly ambiguous and depends on parameter calibration as well as substitution, income and wealth effects, respectively.

<sup>4</sup>A CES utility function could be specified and experiments be made with alternative values for the elasticity of substitution.

<sup>5</sup>An explicit formulation of the optimisation of parents' utility over their own consumption and that of their children is not necessarily important. This is because the optimisation problem would merely relate first period consumption of the household to the weight that parents assign to consumption of their children in utility. The childhood period is conceptually necessary in this model, though, in order to study a change in fertility in period  $t - 1$  that affects the size of the labour force in period  $t$ . This relation can be shown to enter into lifetime utility as a weight on first period consumption,  $\rho_{1(b_t)} > 0$ , that depends positively on the number of children, see Jensen and Jørgensen (2008). We assume, however, that a 1% increase in fertility would increase  $\rho_{1(b_t)}$  by 1%, because parents need to provide more consumption to more children in the household.

<sup>6</sup>Both Chakraborty (2004) and Bohn (2001) have incorporated the length of the retirement period into the utility function, but neither have incorporated the length of the working period.

which are all composed by full leisure, will be substituted by sub-periods that consist of both labour and leisure in the working period. This has a negative impact on lifetime leisure. A novelty of our approach is to scale leisure by  $\chi_t$  to account for this effect. As a result, individuals can now account for the disutility of a fall in lifetime leisure, in case the retirement age should increase, by increasing leisure in their working period. In this case, the effective labour supply would initially rise by the full amount of the increase in the retirement age. But this effect will be counteracted if the disutility of less lifetime leisure induces workers to supply labour less intensively<sup>7</sup>.

The restrictions on  $c_{1t}$  and  $c_{2t+1}$  are presented in (3) and (4),

$$\chi_t c_{1t} = (1 - \kappa_t)(1 - l_t) \chi_t w_t - S_t \quad (3)$$

$$c_{2t+1} = \frac{R_{t+1}}{\lambda_{t+1}} S_t + \gamma_{t+1} (1 - l_{t+1}) \chi_{t+1} w_{t+1} \quad (4)$$

where  $\kappa_t$  is the pension contribution rate,  $S_t$  is the level of savings, and  $\gamma_t$  is the pension replacement rate. In terms of income in the working period,  $w_t \chi_t$ , the wage rate in each sub-period (say, in each year) is denoted by  $w_t$ , while  $\chi_t$  denotes how many sub-periods people have to work (say, the length of the working period in terms of years)<sup>8</sup>.

The gross return to the savings of retirees,  $R_t = (1 + r_t)$ , is scaled by  $\lambda$  to account for the fact that savings must be spread across a given length of the retirement period. In that way, if the retirement age increases so the retirement period is residually reduced, there will be more second period income in each sub-period in retirement. In Bohn (2001),  $\lambda$  does not depend negatively to the retirement age, and in Chakraborty (2004),  $\lambda$  is endogenous to health expenditure and is incorporated so it encompasses both the discount rate and at the same time the length of total life. In our paper, however,  $\lambda$  is also endogenous, but it depends on changes in the retirement age or changes in the total length of adult life, i.e.  $\lambda = \phi - \chi$ . Changes in  $\chi$  is therefore seen to affect  $\lambda$ , and that could not be analysed by neither Bohn (2001) nor Chakraborty (2004).

Combining  $c_{1t}$  and  $c_{2t+1}$  over  $S_t$  yields the intertemporal budget constraint:

$$\chi_t c_{1t} + \frac{\lambda_{t+1}}{R_{t+1}} c_{2t+1} + (1 - \kappa_t) w_t \chi_t l_t = (1 - \kappa_t) w_t \chi_t + \frac{\lambda_{t+1}}{R_{t+1}} \gamma_{t+1} w_{t+1} \chi_{t+1} \quad (5)$$

Note the roles of  $\chi$  and  $\lambda$  as implicit prices on consumption and leisure: consumption and leisure must be spread across the lengths of working and retirement periods, respectively. Utility is therefore increasing in  $\chi$  and  $\lambda$ , but so are the implicit prices on consumption and leisure.

By maximising lifetime utility (2) subject to the intertemporal budget constraint (5), two first order conditions are derived: first, the Euler equation in (6),

<sup>7</sup>By modelling the utility of leisure in this way we implicitly add the value of second period leisure into the utility function without having to maximise explicitly with respect to  $l_{t+1}$ .

<sup>8</sup>If the retirement age increases, and the capital-labour ratio and the wage rate fall, then the income of workers may either increase or decrease depending on whether the drop in the wage rate across all sub-periods accounts for smaller fall in income than the increase in income induced by the additional sub-periods of work.

$$c_{1t} = \left( \frac{\rho_{1(b_t)}}{\rho_2} \right) E_t \left\{ \frac{c_{2t+1}}{R_{t+1}} \right\} \quad (6)$$

and, second, the optimality condition for first period consumption and leisure in (7):

$$l_t = \left( \frac{\beta}{\rho_{1(b_t)}} \right) \frac{c_{1t}}{(1 - \kappa_t) w_t} \quad (7)$$

Note, that households prioritise less consumption in the first period if fertility decreases. This will lower  $\rho_{1(b_t)}$  in (6) and (7), such that  $c_{1t}$  falls relative to  $c_{2t+1}$  and  $l_t$ . If  $\chi$  or  $\lambda$  changes, the optimality conditions will remain unaffected<sup>9</sup>.

### 2.3 Social Security

The economy is assumed to operate with a PAYG pension system, given by the following identity,

$$\lambda_t \gamma_t u_t w_t N_{t-1}^w = \kappa_t u_t w_t N_t^w \quad (8)$$

where the left (right) hand side illustrates the pension benefits (contributions). Neither  $\kappa$  nor  $\gamma$  need to be fixed, so the PAYG system can in principle display either defined benefits (DB) or defined contributions (DC) schemes. To reflect the empirical fact that the DB system is the most widespread PAYG arrangement (Gruber and Wise, 1999), we assume that benefits are held constant whereas the contribution rate may vary<sup>10</sup>:

$$\kappa_t = \gamma \left( \frac{\phi_t - \chi_{t-1}}{1 + n_t^w} \right) \quad (9)$$

### 2.4 Technology and resources

Output,  $Y_t$ , is assumed to be produced by firms with a Cobb-Douglas technology in terms of capital,  $K_t$ , and labour:

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$$

Productivity is denoted by  $A_t$  and is assumed to be stochastic and growing at a rate,  $a_t$ , such that  $A_t = (1 + a_t) A_{t-1}$ , where  $a_t$  is assumed identically and independently distributed. The return to capital and the wage rate are standard and defined by  $r_t(K_t) = f'(K_t)$  and  $W_t(K_t) = f(K_t) - K_t f'(K_t)$ , and  $k_{t-1} \equiv K_t / (A_{t-1} L_{t-1})$  defines the capital-labour ratio over growth rates.<sup>11</sup> By assuming that firms are identical, capital will be accumulated through the savings of workers, i.e.  $K_{t+1} =$

<sup>9</sup>The increase in utility of a longer working or retirement period is offset by a corresponding increase in the implicit prices of consumption and leisure in the intertemporal budget constraint.

<sup>10</sup>Evidently, if the longevity of current retirees increases, the retirement period would residually increase, given that the retirement age remains unchanged, and this would call for a higher contribution rate. Similarly, an increase in the retirement age, given an unchanged length of life, would yield a lower contribution rate. Last, but not least, if fertility falls so will the growth in the number of workers and contributions need to rise to balance the PAYG budget.

<sup>11</sup>Since a smaller labour force leads to an increase in the capital-labour ratio, changes in factor returns are likely to occur, see Kotlikoff et al. (2001), Murphy and Welch (1992) and Welch (1979).

$N_t^w S_t$ . Furthermore, we assume that over one generational period (app. 30 years) capital fully depreciates. The constraint on the economy's aggregate resources is,

$$Y_t - K_{t+1} = \chi_t N_t^w c_{1t} + \lambda_t N_{t-1}^w c_{2t} \quad (10)$$

which features the lengths of the working and retirement periods, respective, in connection with the sub-period rates of consumption. This completes the outline of the model. Next, we present our solution method.

## 2.5 Solving the model

We solve the model analytically for the responses of economic variables to changes in fertility and the statutory retirement age. The solution method is designed to provide analytical elasticities of economic variables with respect to stochastic shocks, and it involves transforming the stochastic OLG model into a version that is log-linearised around the steady state of the model. Our analytical approach facilitates the isolation of the necessary response of the statutory retirement age that will offset any negative responses of labour supply<sup>12</sup>.

A version of the method of undetermined coefficients, which relies on Uhlig (1999) and extended by Jørgensen (2008), is adopted to obtain the analytical solution for the recursive equilibrium law of motion. The variables of the linearised model are stated in efficiency units and in terms of percentage deviations from the steady state (marked with "hats")<sup>13</sup>. A linear law of motion for the recursive equilibrium of the economy is conjectured,

$$\begin{aligned} \hat{x}_t &= \mathbf{P}\hat{x}_{t-1} + \mathbf{Q}\hat{z}_t \\ \hat{v}_t &= \mathbf{R}\hat{x}_{t-1} + \mathbf{S}\hat{z}_t \end{aligned}$$

which is characterised by linear relationships between endogenous state variables in the vector  $\hat{x}_t$  and exogenous state variables (the shocks) in the vector  $\hat{z}_t$ . The non-state endogenous (jump) variables are denoted by  $\hat{v}_t$ . The coefficients in the matrices  $\mathbf{P}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$ , and  $\mathbf{S}$  are interpreted as elasticities.

As an example of how a given endogenous variable is determined by changes in e.g. lagged fertility,  $\hat{b}_{t-1}$ , or the statutory retirement age,  $\hat{\chi}_t$ , we illustrate the law of motion for leisure,

$$\hat{l}_t = \pi_{lk}\hat{k}_{t-1} + \pi_{lc2}\hat{c}_{2t-1} + \pi_{lb1}\hat{b}_{t-1} + \pi_{l\chi}\hat{\chi}_t \quad (11)$$

where, e.g.,  $\pi_{l\chi}$  denotes the elasticity ( $\pi$ ) of leisure ( $l$ ) with respect to the retirement age ( $\chi$ )<sup>14</sup>.

A key advantage of this analytical approach is that the impact on leisure of a change in the retirement age is stated in terms of an elasticity,  $\pi_{l\chi}$ , the size of

<sup>12</sup>The advantage of an analytical, closed form, solution is that changes in any economic variable can be traced back to the underlying parameters and fundamental properties of the model. Thereby, valuable intuition on the impact of falling fertility on economic variables can be gained.

<sup>13</sup>See Appendix A for more details on the solution technique.

<sup>14</sup>All endogenous variables  $\{\hat{k}_t, \hat{c}_{1t}, \hat{c}_{2t}, \hat{l}_t, \hat{y}_t, \hat{R}_t, \hat{w}_t, \hat{\kappa}_t\}$  can be expressed in this fashion. The complete vector of exogenous state variables is  $\hat{z}_t \in \{\hat{\chi}_{t-1}, \hat{\chi}_t, \hat{a}_t, \hat{b}_{t-1}, \hat{b}_t, \hat{\phi}_{t-1}^e, \hat{\phi}_t^e, \hat{\phi}_t^u\}$ , but (11) only illustrates the shocks to lagged fertility and the statutory retirement age. The vector of endogenous state variables is  $\{\hat{k}_t, \hat{c}_{2t}\}$  so these remain in equation (11) no matter which shocks are examined.



which, *by construction*, assumes a 1% shock to the statutory retirement age,  $\widehat{\chi}_t$ . Therefore, we simply ask: "how will leisure change if there was suddenly an increase in the statutory retirement age of 1%?". Using this terminology, we basically make comparative statics with a model that is otherwise designed to be stochastic<sup>15</sup>. This procedure is, by now, standard in the real-business-cycle literature (see, e.g., Uhlig, 1999). Our contribution, in this context, is to tailor the method in Uhlig (1999) to fit a stochastic OLG model, which is complicated by changes in the retirement age that implies future changes in length of the retirement period.

The elasticities can be interpreted (both analytically and numerically) and employed in connection with the design of policy rules for the retirement age when fertility has fallen and brought down the size of the labour force. We calibrate the analytical expressions of the model with values, in table 1, that we trust are realistic, and subsequently derive the numerical elasticities of the model. Importantly, we make robustness analyses with the weight on leisure in the utility function in section 4, since the model predictions depend crucially on the calibration of this parameter.

Parameter	Value	Interpretation of steady state parameters
$\alpha$	1/3	The capital share in output
$\gamma$	0.35	The pension replacement rate <sup>16</sup>
$a$	0.40	The steady state growth rate of productivity
$\delta$	1	The rate of capital depreciation
$\chi$	1	The length of the working period
$\lambda$	1	The length of the retirement period
$b$	0.1	The rate of growth in the number of children
$\beta$	1	The weight on leisure in the utility function
$\pi_{\rho_1(b)}$	1	The elasticity on the weight of first period consumption in utility with respect to the birth rate
$\rho_2$	0.292	The consumption discount rate <sup>17</sup>

Table 1. Parameter calibration

### 3 Equilibrium with shocks

How will a fall in fertility affect the effective supply of labour? Obviously, the size of the labour force falls with the decline in past fertility. Labour may be supplied by households more or less intensively, though, when fertility has fallen. In addition, if the retirement age rises in response to a shrinking labour force this, in itself, may also lead to a more or less intensive supply of labour.

Two shocks are examined, namely an exogenous shock to the lagged birth rate,  $\widehat{b}_{t-1}$ , and an exogenous change in the retirement age,  $\widehat{\chi}_t$ . We then derive the elasticities of economic variables with respect to these two shocks. In this section we assume that changes in the statutory retirement age are outside government control – as if the changes were stochastic. This provides insights as to how the economy and

<sup>15</sup>Note that the size of a stochastic shock to, e.g., fertility could be any value from a given pre-specified distribution of innovations.

<sup>16</sup>The payroll tax rate will then be  $\kappa = \gamma(\phi - \chi) / (1 + n^w) = 0.30$ .

<sup>17</sup>The calibration of the discount rate equals 0.960 per year or 0.292 over a 30 year period, and generates a savings rate of 20%.

intergenerational distributions of welfare are affected by demographic shocks under a *passive* policy framework. In section 4, however, the statutory retirement age is assumed to be under government control and used as a policy instrument<sup>18</sup>. We limit the focus on leisure ( $\widehat{l}_t$ ) and consumption ( $\widehat{c}_{1t}$ ) for workers, as well as retirees' consumption ( $\widehat{c}_{2t}$ ).

### 3.1 Effects of low fertility

In this section we analyse the macroeconomic impacts of the significant historical declines in fertility rates. A lagged shock to fertility is therefore the focus. The economy is represented by a linear law of motion in terms of elasticities for endogenous variables with respect to a *positive* fertility shock. These elasticities are reported in table 2. The relevance of decomposing the net effect on each variable into various sub-effects is to get a better understanding of the magnitudes involved in the numerical simulations.

Table 2. A fall in fertility

Variable	Value	Elasticity
$\pi_{c1b1} =$	-0.02	$= [\pi_{c2k} - \pi_{Rk}] \pi_{kb1}$
$\pi_{lb1} =$	-0.11	$= \pi_{c1b1} + \Lambda_{23}\pi_{\kappa b1} - \Lambda_{22}\pi_{wb1}$
$\pi_{c2b1} =$	0.54	$= [\Lambda_{15}\pi_{lb1} - \Lambda_3\pi_{c1b1} - \Lambda_5\pi_{kb1} - \Lambda_2] / \Lambda_4$
$\pi_{kb1} =$	-0.02	$= \frac{\Lambda_{12}\pi_{wb1} - \Lambda_{21}\pi_{lb1} - \Lambda_8\pi_{\kappa b1}}{\Lambda_9\pi_{wk} - \Lambda_7\pi_{c2k} + \Lambda_{12}\pi_{Rk} - \Lambda_{20}\pi_{lk}}$

The key issue is how work-leisure choices will be determined subsequent to the fertility decline. This result is subject to a number of counteracting effects and remains theoretically ambiguous. Our simulations imply, however, that leisure will *increase* by 0.11% after a 1% fertility fall<sup>19</sup>. The increase in leisure corresponds to a reduction in the intensity of labour supply, which will magnify the initial fertility-induced effect on the shrinking effective labour supply and the increasing capital-labour ratio.

Changes in wages and pension contributions basically determine the effects on workers' consumption after the shock to fertility (see Jensen and Jørgensen, 2008). On the other hand, since labour supply is a choice-variable, consumption and leisure are interrelated and indirectly affect the capital-labour ratio: more leisure leads to an even higher capital-labour ratio, higher wages, and lower capital returns (see figures 2a and 2b). Therefore, by examining the intertemporal budget constraint in (5) we can analyse the substitution, income, and wealth effects on leisure<sup>20</sup>.

<sup>18</sup>In the present section, we are interested *only* in the effects of a change in the retirement age and not in what *causes* that change. It is often more natural to think of a shock to the statutory retirement age as exogenous to the agent and controlled by the government.

<sup>19</sup>Elasticities are, by construction, derived for a positive 1% shock to fertility. Therefore, the elasticities of economic variables with respect to a *negative* fertility shock must be interpreted with the opposite sign of those displayed in table 2.

<sup>20</sup>In the case where labour supply is exogenous (see, e.g., Jensen and Jørgensen, 2008), the only effect on the capital-labour ratio originates from the lower fertility rate.

Figure 2a. Wage rate

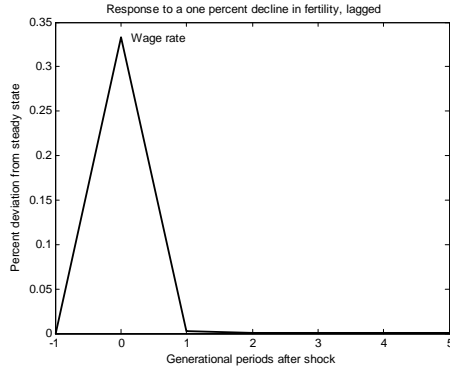


Figure 2b. Return to capital

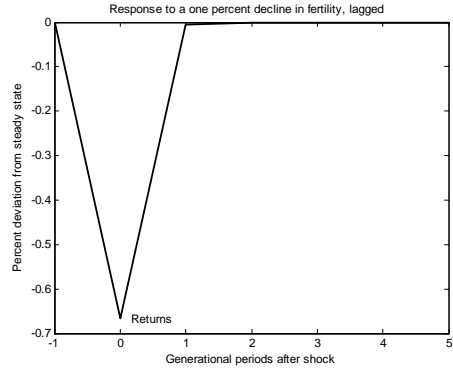


Figure 2c. Workers' leisure

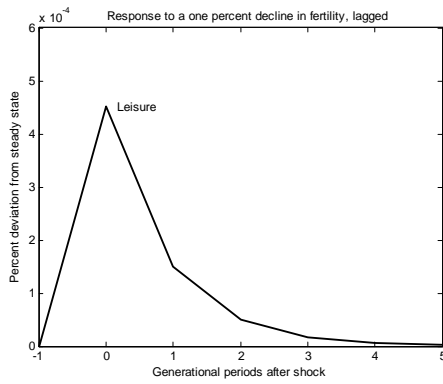
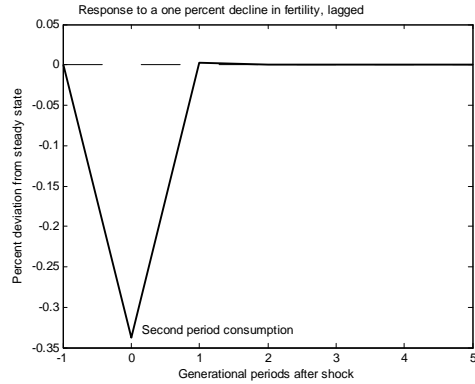


Figure 2d. Retirees' consumption



The substitution effect on leisure comes from a shrinking labour force that alters factor payments: wages increase; the return to capital falls. The price (opportunity cost) of leisure thus increases so the substitution effect on leisure is negative. A given level of income can now buy less resulting in negative income effects on all goods, including leisure. The wealth effect is positive for all goods, because the increased wage rate appears in *lifetime* income<sup>21</sup>. The dynamics of leisure, as well as retirees' consumption, are illustrated by the simulated trajectories in figures 2c and 2d, respectively<sup>22</sup>.

In an influential paper, Weil (2006) finds that a key mechanism through which aggregate income and welfare are affected by population ageing is the distortion from taxes to fund PAYG pension systems. This mechanism is also present here: the price on leisure depends on  $\kappa$ , i.e. the (flat) PAYG contribution rate. With labour supply being endogenous, this distorting tax rate implies that the positive wealth effect will more than offset the (negative) sum of substitution and income effects (i.e.  $\pi_{lb1} = 0.11$ )<sup>23</sup>.

<sup>21</sup>See the right-hand side of the intertemporal budget constraint in equation (5).

<sup>22</sup>The dynamics of first-period consumption is identical to the simulated trajectory for leisure, though larger numerically.

<sup>23</sup>The distorting effects increase with the size of the pension system, so the larger  $\kappa$  is the larger is, the larger is  $\pi_{lb1}$ . If taxation was lump sum and not distortionary these three effects will offset each other so the net effect on leisure is zero, given that intertemporal elasticity of substitution equal to one, as in our case.

There are additional effects to consider in order to obtain a complete analysis of the impacts of low fertility. Due to, first, a changing capital-labour ratio and, second, the presence of distortionary taxation we have to consider "factor price effects" and "fiscal effects", respectively: a negative fertility shock implies that each worker (in the smaller labour force) must pay more taxes (because the benefits to retirees are assumed fixed in a DB system). Thus the fiscal effect is negative. In addition, workers will receive higher wages due to the higher capital-labour ratio so the factor price effect is positive. This net effect is caused by a *direct* effect and an *indirect* effect: The population growth rate  $(1 + n_t^w)$  falls which directly reduces the size of the labour force. The indirect effect is due to the endogenous response of leisure ( $\pi_{lb1} > 0$ ) which has a reinforcing negative effect on labour supply. The implication for effective labour supply is, therefore, that the initial negative effect from lower fertility is amplified by lower intensity of labour supply due to the demand for more leisure as a consumption-equivalent good.

The net effect on consumption is consequently ambiguous, but our simulations show that consumption increases for a negative fertility shock:  $\pi_{c1b1} = 0.02$  and  $\pi_{c2b1} = -0.54$ , such that workers gain in terms of consumption and leisure and retirees lose in terms of consumption. Thus, there will be an uneven intergenerational distribution of the economic effects. While such welfare implications will not be pursued further in this paper, it is an interesting topic for future research.

### 3.2 Effects of a higher statutory retirement age

The statutory retirement age can be used as a policy instrument to increase effective labour supply by retaining workers in the labour force for a longer period of time and denying them PAYG pension benefits until this later date. Such changes will have economic implications that should be well understood by policy makers before designing a policy rule for the retirement age. The purpose of this section is to present a *positive* analysis on how changes in the retirement age affect key economic variables. We are only interested in the effects of an *exogenous* change in the retirement age and not in what causes the change<sup>24</sup>. When we are well informed about the implications of a change in the retirement age we move on, in the next section, to present a *normative* analysis of the retirement age – based on the assumption that the statutory retirement age is under government control.

An increase in the retirement age will tend to directly increase labour supply and lower the length of the retirement period, which is in line with our specification of the length of the retirement period is residually determined by the length of the working period ( $\lambda = \phi - \chi$ ). As a result, workers need to save less for a shorter retirement period.

The change in leisure is determined through the same channels as a fertility shock: the substitution effect, the income effect, the wealth effect, and the fiscal

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<sup>24</sup>While we assume, in this section, that the *statutory* retirement age is exogenous to the consumer, this will not be the case for the *effective* retirement age, because the intensity of labour supply is endogenous to the household in this paper. If the statutory retirement age increases, no matter why, households may decide to supply less labour. If one assumes that this reduction in labour supply takes place towards the end of households' working life (rather than being spread across all sub-periods of the working period), the reduction reflects the fact that people may retire earlier based on their own savings and thus represent a *fall* in the effective retirement age.

and factor price effects, respectively. These dynamics are all intertwined through both exogenous and endogenous changes in the capital-labour ratio and changes in the pension contributions and benefits. The net effect on the capital-labour ratio is negative if the intensity of labour supply does not endogenously fall more than the retirement age has increased. In that case the net effect on capital returns remains positive and the wage rate will fall. This will indeed be the case since the effective labour supply increases by 0.96% because leisure increases by  $\pi_{l\chi} = 0.04\%$  for each 1%–increase in the retirement age increases (see table 3)<sup>25</sup>.

Variable	Value	Elasticity
$\pi_{c1\chi} =$	1.06	$= [\pi_{c2k} - \pi_{Rk}] \pi_{k\chi} + (\pi_{c2\chi1} - \pi_{R\chi1})$
$\pi_{l\chi} =$	0.04	$= \frac{[\pi_{c2k} - \pi_{Rk}] \pi_{k\chi} + (\pi_{c2\chi1} - \pi_{R\chi1}) + (\Lambda_{23} \pi_{\kappa\chi} + \Lambda_{22} \Lambda_{11})}{1 + \Lambda_{22} \Lambda_{11}}$
$\pi_{c2\chi} =$	0.42	$= [\Lambda_9 \pi_{wk} - \Lambda_7 \pi_{c2k} + \Lambda_{12} \pi_{Rk} - \Lambda_{20} \pi_{lk}] \pi_{k\chi}$ $- \Lambda_{21} \pi_{l\chi} + \Lambda_{12} \pi_{w\chi} - \Lambda_8 \pi_{\kappa\chi}$
$\pi_{k\chi} =$	-0.31	$= \frac{\Lambda_{15} \pi_{l\chi} - \Lambda_3 \pi_{c1\chi} - \Lambda_4 \pi_{c2\chi} - \Lambda_2}{\Lambda_5}$

Regarding the fiscal effect: workers now face more subperiods during which they work and has to contribute to the fixed PAYG benefits of retirees. This implies less need for savings to finance a shorter retirement period, so workers save less and free resources for leisure and first-period consumption. Thus, a positive fiscal effect.

In terms of substitution, income and wealth effects on leisure, we find that the substitution effect is negative due to the net increase in the price on leisure. The dynamics of factor payments therefore generates a positive wealth effect (lifetime income increases disproportionately to the fall in the wage rate but proportionally to the increase in the statutory retirement age) and a negative income effect (an unchanged level of income can buy less consumption and leisure since leisure has become more expensive). The positive wealth effect offsets the negative sum of substitution and income effects, partly due to distortionary taxation, so the effect on leisure is positive<sup>26</sup>.

A particularly important mechanism in this model is that we account for the disutility of work in terms of less lifetime leisure when the retirement age increases, i.e. workers will be induced to supply labour less intensively when the sub-periods of full leisure in retirement are reduced.

<sup>25</sup>This is also confirmed by the elasticities of the wage rate and capital returns with respect to the retirement age ( $\pi_{w\chi} = \alpha(1 - \pi_{l\chi}) = -0.32$ ;  $\pi_{R\chi} = (1 - \alpha)(1 - \pi_{l\chi}) = 0.64$ ) which represents a negative (positive) factor price effects for workers (retirees). The *direct* effect on, e.g., capital returns is  $(1 - \alpha)$  due to the fall in the capital-labour ratio, while the *indirect* effect originating from endogenous labour supply is  $(1 - \pi_{l\chi})$ .

<sup>26</sup>As a result of the dynamics above, workers receive a lower wage rate over a longer working period, which renders the net impact on first-period consumption theoretically ambiguous. We find that  $\pi_{c1\chi} = 1.06$  is positive, however, and that it depends, especially, on the need for less savings to finance a shorter retirement period and a higher lifetime income due to more sub-periods of work. Retirees tend to gain in terms of consumption. The net effect is ambiguous, but our simulations show an increase in  $\pi_{c2\chi}$ .

An increase in the retirement age does not yield an equal increase in effective labour supply when fertility has declined. This complicates the analysis of an offsetting policy rule for the statutory retirement age. That is precisely why it is crucial to emphasize the dynamics of the *intensive* margin of labour supply relative to the *extensive* margin. This is the topic of the next section.

## 4 Policy Reform

We have seen that three main forces are operating when fertility or the statutory retirement age change: the factor price effect; the fiscal effect; and the endogenous intensity of labour supply (determined, in turn, by substitution, income and wealth effects). In this section, we make use of our general equilibrium framework to derive *how much the statutory retirement age should increase in order to offset the decline in the labour force caused by low fertility in the past*<sup>27</sup>. It is important, though, which role one assigns to the statutory retirement age, and we operate under the explicit assumption that the retirement age is an exogenous variable that is under government control. Note, that our analyses are independent of the social desirability of any intergenerational (welfare) distribution of the associated effects<sup>28</sup>.

The effective labour supply comprises three elements: first, the fertility rate,  $\widehat{b}_{t-1}$ ; second, the extensive margin limited by the retirement age,  $\widehat{\chi}_t$ ; and third, the intensity with which workers work (the intensive margin,  $\widehat{u}_t = -\widehat{l}_t$ ). The effective labour supply is  $d_t = (1 + n_t^w)(1 - l_t)$ , or in log-deviations from steady state:

$$\widehat{d}_t = \widehat{b}_{t-1} + \widehat{\chi}_t - \widehat{l}_t \quad (12)$$

Assume first that the intensity of labour supply is exogenous and that we examine a 1% decline in fertility. It is then clear from (12) that the necessary response of the statutory retirement age, which would offset the fertility decline, i.e.  $\widehat{d}_t = \widehat{\chi}_t + \widehat{b}_{t-1} \equiv 0$ , would just be a proportional increase of  $\widehat{\chi}_t = 1\%$ . However, if the intensity of labour supply is in fact endogenous, so  $\widehat{l}_t \neq 0$ , then clearly the response of  $\widehat{\chi}_t$  would have to be different from 1%. In our case, the initial effect from the fertility decline on the effective labour supply will be reinforced because leisure increases, so the statutory retirement age would have to increase even more than 1%. To derive the offsetting response of  $\widehat{\chi}_t$  we insert the linear law of motion for  $\widehat{l}_t$  to obtain:

$$\widehat{\chi}_t = [\pi_{lb1}\widehat{b}_{t-1} + \pi_{l\chi}\widehat{\chi}_t] - \widehat{b}_{t-1} \quad (13)$$

From (13) isolate  $\widehat{\chi}_t$ , and insert the numerical elasticities,  $\pi_{lb1}$  and  $\pi_{l\chi}$ , and the negative fertility shock,  $\widehat{b}_{t-1} = -1$ :

$$\widehat{\chi}_t = - \left[ \frac{1 - \pi_{lb1}}{1 - \pi_{l\chi}} \right] \widehat{b}_{t-1} = 1.15 \quad (14)$$

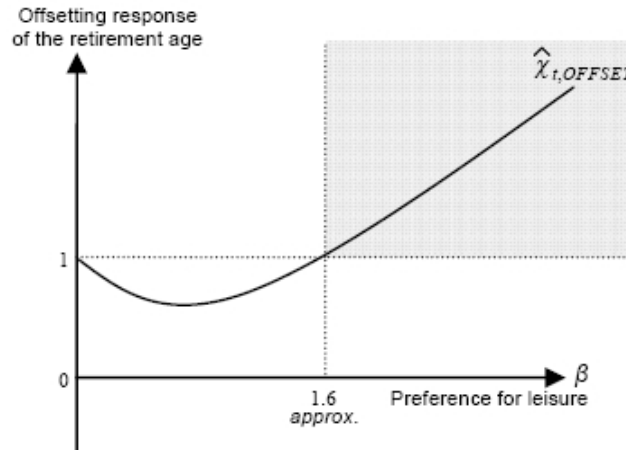
<sup>27</sup>Proposals for using the retirement age as a policy instrument are found in, e.g., de la Croix et al. (2004) and Andersen, Jensen and Pedersen (2008). Also, Cutler (2001) recommends an extension of Bohn (2001) to incorporate "the length of the period where people work".

<sup>28</sup>Jensen and Jørgensen (2008) evaluates the attractiveness of an uneven distribution of the economic effects associated with low fertility in a model with exogenous labour supply, while Jørgensen (2008) does so in a model with endogenous labour supply.

Observe that if  $\pi_{b1} < \pi_{l\chi}$  the optimal response is  $\hat{\chi}_t > 1$ . So, we conclude that the statutory retirement age has to increase more than fertility fell in order to offset the negative impact on the effective labour force. The offsetting response of the statutory retirement age, when  $\hat{b}_{t-1} = -1\%$  and the weight on leisure in utility is  $\beta = 3$ , amounts to  $\hat{\chi}_t = 1.15\%$ .

These dynamics are due to the choice of leisure by individuals, which will increase both when fertility falls and when the statutory retirement age increases. Thus, the negative fertility-impact on labour supply is amplified. Since the weight that households place on leisure is so crucial to the macroeconomic dynamics when the labour force shrinks, this weight should be tested for alternative values. The literature suggests various values for  $\beta$  generally within the range  $\beta \in \{1; 9\}$  (see, e.g., Blackburn and Cipirani, 2002; Cardia, 1997; Chari et al., 2000; Jonsson, 2007). We have calibrated our model with  $\beta = 3$ , as an example, and found the offsetting response of the statutory retirement age to be larger than the fertility rate ( $\hat{\chi}_t = 1.15$ ). In terms of robustness analysis, however, we simulate the value for  $\hat{\chi}_t$  given alternative values for  $\beta$  and illustrate the results in figure 3.

Figure 3. Robustness analysis



For  $\beta = 0$ , the analysis for the offsetting response of  $\hat{\chi}_t$  corresponds to the exogenous labour supply scenario. The 1% fall in fertility can therefore be exactly offset by a 1% increase in the statutory retirement age. For small values of  $\beta$  there is a tendency for the offsetting response of the statutory retirement age to be even less than the fertility-induced fall in labour supply. This means that a contraction in the labour force combined with an increase in the statutory retirement age increases the intensity of labour supply (reduces leisure). The large (net) increase in the price on leisure,  $(1 - \kappa) w\chi$ , when fertility falls and the statutory retirement age increases, drives the substitution and income effects to outweigh the wealth effect so the intensity of labour supply increases. As the weight on leisure increases beyond app. 1.6 this trend is reversed. Households now value leisure to such a high extent that substitution and income effects no longer dominate the decision to "purchase" leisure. The higher the preference for leisure the greater the tendency

to substitute for leisure, and this trend exerts downward pressure on the intensity of labour supply. As a result, the offsetting response of the statutory retirement age becomes increasingly larger than the fall in fertility (the grey area in figure 3).

An important question now arises: what is the empirical trend in the preference for leisure? If households over the past decades have had a tendency to substitute for more leisure as real wages (and, thus, the price on leisure) have increased, then the offsetting response of the statutory retirement age is likely to equal a value on the curve in the grey area of figure 3. In that case, policy makers should take the resulting dynamics into account when designing policy rules for the retirement age in order to overcome the problems for welfare arrangement when fertility, and thus, labour supply has fallen.

According to Pencavel (1986), the share of life that men spend at work for pay has fallen significantly. In fact, workers are retiring from the labour force at younger ages, the number of hours worked per day or per week has fallen, and the number of holidays has increased - and holidays have become longer. Schmidt-Sørensen (1983) finds for Denmark that the number of working hours per *week* fell by 25% over the period 1911-83, and by 15% over the period 1955-83. Similarly, the number of working hours per *year* fell by 34% over the period 1911-81.

While the fraction of lifetime spent at market work may also have fallen because more time has been allocated to human capital investment, by spending more years within the educational system, the empirical evidence clearly suggests that the preference for leisure has been increasing for decades. It is therefore likely that the dynamics of the economy, when facing a shrinking labour force, will generate more demand for leisure as real wages increase. This implies that the offsetting response of the labour force will be in a *more* than 1:1 relationship to the contraction in the labour force. A model which does not incorporate labour supply as a choice variable may fail to capture some important macroeconomic dynamics. The ability to analyse the impacts of shrinking labour forces for various values for the preference for leisure thus marks a significant extension of the framework used by, e.g., Bohn (2001). Such an analysis would not be feasible without the explicit relationship in the model between the extensive and intensive margins of labour supply.

## 5 Conclusion

This paper has developed an intertemporal setting in which retirement policy can be used to correct for fertility-induced changes in the supply of labour. Our main finding is that the retirement age should increase *more* than proportionately to a fertility decline in order to account for negative responses of the intensity of labour supply. However, this result depends crucially on the preference for leisure by households. In line with empirical evidence there has been a tendency for leisure to rise when real wages increase. And real wages tend to increase when labour supply shrinks as a result of a fertility decline. Therefore, the necessary offsetting response of the statutory retirement age is likely to be even higher than previously believed. Without an analytical framework linking the endogenous intensive margin to the extensive margin of labour supply, this analysis would not be feasible.

An additional finding is that leisure may increase when the statutory retirement



age increases. This could be interpreted as an endogenous drop in the voluntary early retirement age, financed by workers' own savings. This is exactly the opposite of what is intended by the policy rule of increasing the statutory retirement age. This counteracting mechanism is part of the underlying reason why we derive a more-than-proportionate offsetting increase in the statutory retirement age.

The analytical framework is subject to a number of limitations. The utility function has been modelled in accordance with our best beliefs of how to incorporate the value of leisure and the lengths of periods. However, the robustness of our result could be examined in greater detail for alternative specifications of the utility function: for example, by adopting a CES specification that allows for robustness analyses with respect to the elasticity of substitution. In addition, we assume that the economic impacts of changes in dependency ratios can be analysed in a linearised model. Simulation exercises with CGE models should, in the future, be performed to yield a more empirically accurate, and country-specific, foundation for designing a policy rule for the retirement age. Last but not least, human capital accumulation may have the implication that workers choose to invest in education to a higher extent when fertility is low because they receive higher wages. As a result, the supply of labour may incorporate a higher productivity. Thus, there may be less need for the statutory retirement age to increase to completely offset the smaller labour force. These issues may modify our results, and are interesting subjects for future research.

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## A The solution method

The way we apply the method of undetermined coefficients relies on Uhlig (1999). The method is adapted, though, to the stochastic OLG structure of our model in line with Jørgensen (2008). This appendix provides a brief overview of the solution method, but we refer to the aforementioned authors for more details. All endogenous variables from the log-linearised model,  $\widehat{e}_t \in \{\widehat{k}_t, \widehat{c}_{2t}, \widehat{c}_{1t}, \widehat{l}_t, \widehat{y}_t, \widehat{R}_t, \widehat{w}_t, \widehat{\kappa}_t\}$ , are written as linear functions of a vector of endogenous and exogenous state variables, respectively. The vector of endogenous state variables is  $\widehat{x}_t \in \{\widehat{k}_t, \widehat{c}_{2t}\}$  of size  $m \times 1$ <sup>29</sup>, the vector of endogenous non-state variables is  $\widehat{v}_t \in \{\widehat{c}_{1t}, \widehat{l}_t, \widehat{y}_t, \widehat{R}_t, \widehat{w}_t, \widehat{\kappa}_t\}$  of size  $j \times 1$ , while the vector of exogenous state variables is  $\widehat{z}_t \in \{\widehat{\chi}_{t-1}, \widehat{\lambda}_t, \widehat{a}_t, \widehat{b}_{t-1}, \widehat{b}_t, \widehat{\phi}_{t-1}^e, \widehat{\phi}_t^e, \widehat{\phi}_t^u\}$  of size  $g \times 1$ . The log-linearised equations are in written matrix notation in the following equilibrium relationships,

$$0 = \mathbf{A}\widehat{x}_t + \mathbf{B}\widehat{x}_{t-1} + \mathbf{C}\widehat{v}_t + \mathbf{D}\widehat{z}_t \quad (15)$$

$$0 = E_t [\mathbf{F}\widehat{x}_{t+1} + \mathbf{G}\widehat{x}_t + \mathbf{H}\widehat{x}_{t-1} + \mathbf{J}\widehat{v}_{t+1} + \mathbf{K}\widehat{v}_t + \mathbf{L}\widehat{z}_{t+1} + \mathbf{M}\widehat{z}_t] \quad (16)$$

$$\widehat{z}_{t+1} = \mathbf{N}\widehat{z}_t + \varepsilon_{t+1}, \quad E_t [\varepsilon_{t+1}] = 0 \quad (17)$$

where  $\mathbf{C}$  is of size  $h \times j$ , where  $h$  denotes the number of non-expectational equations. In this particular OLG model  $h = j$ , due to the definition of  $\widehat{x}_t = \{\widehat{k}_t, \widehat{c}_{2t}\}$ , because with merely the capital stock as a state variable  $h < j$ , and the system cannot not be solved<sup>30</sup>. The matrix  $\mathbf{F}$  is of size  $(m + j - h) \times j$ , and it is assumed that  $\mathbf{N}$  has only stable eigenvalues.

The recursive equilibrium is characterized by a conjectured linear law of motion between endogenous variables in the vector  $\widehat{e}_t$ , and state variables (endogenous and exogenous, respectively) in the vectors  $\widehat{v}_t$  and  $\widehat{z}_t$ . The conjectured linear law of motion is written as,

$$\widehat{x}_t = \mathbf{P}\widehat{x}_{t-1} + \mathbf{Q}\widehat{z}_t \quad (18)$$

$$\widehat{v}_t = \mathbf{R}\widehat{x}_{t-1} + \mathbf{S}\widehat{z}_t \quad (19)$$

where the coefficients in the matrices  $\mathbf{P}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$ , and  $\mathbf{S}$  are interpreted as elasticities. These linear relationships between endogenous variables and state variables could alternatively be written out for each variable in  $\widehat{e}_t$ , as e.g for leisure,  $\widehat{l}_t$ ,

$$\begin{aligned} \widehat{l}_t &= \pi_{lk}\widehat{k}_{t-1} + \pi_{lc2}\widehat{c}_{2t-1} + \pi_{l\chi1}\widehat{\chi}_{t-1} + \pi_{l\chi}\widehat{\chi}_t + \pi_{la}\widehat{a}_t \\ &\quad + \pi_{lb1}\widehat{b}_{t-1} + \pi_{lb}\widehat{b}_t + \pi_{l\phi e1}\widehat{\phi}_{t-1}^e + \pi_{l\phi e}\widehat{\phi}_t^e + \pi_{l\phi u}\widehat{\phi}_t^u \end{aligned}$$

where e.g.  $\pi_{la}$  denotes the elasticity ( $\pi$ ) of leisure ( $l$ ) with respect to productivity ( $a$ ). The stability of the system is determined by the stability of the matrix  $\mathbf{P}$ , given the assumptions on the matrix  $\mathbf{N}$ .

The stable solution for this system boils down to solving a matrix-quadratic equation in line with Uhlig (1999). The matrix-quadratic equation can be solved as

<sup>29</sup>In order to solve the model it is necessary to have at least as many state variables as there are expectational equations in the model ( $h \geq j$ ).

<sup>30</sup>Note that if  $h > j$  the equations in this section become slightly more complicated, see Uhlig (1999), but a solution is still feasible.

a generalized eigenvalue-eigenvector problem, where the generalized eigenvalue,  $\delta$ , and eigenvector,  $q$ , of matrix  $\Xi$  with respect to  $\Delta$  are defined to satisfy:

$$\delta \Delta q = \Xi q$$

$$0 = (\Xi - \delta \Delta) q$$

For this particular stochastic OLG model  $\Delta$  is invertible so the generalized eigenvalue problem can be reduced to a standard eigenvalue problem of solving instead the expression  $\Delta^{-1}\Xi$  for eigenvalues-eigenvectors, as in (20). Then,  $\Delta^{-1}\Xi$  is diagonalized in (21) since each eigenvalue,  $\delta_i$ , can be associated with a given eigenvector,  $q_m$ .

$$(\Delta^{-1}\Xi - \delta \mathbf{I}) q = 0 \tag{20}$$

$$\mathbf{P} = \Omega \Delta^{-1} \Xi \Omega^{-1} \tag{21}$$

The matrix  $\Delta^{-1}\Xi = \text{diag}(\delta, \dots, \delta_m)$  then contains the set of eigenvalues from which a saddle path stable eigenvalue can be identified, and the matrix  $\Omega = [q_1, \dots, q_m]$  contains the characteristic vectors. Ultimately, the matrix  $\mathbf{P}$ , governing the dynamics of the OLG model, is derived, and the system can be "unfolded" to provide the elasticities in the matrices  $\mathbf{Q}$ ,  $\mathbf{R}$ , and  $\mathbf{S}$ . For more detail on the solution technique for RBC models we refer to Uhlig (1999).